PRINCE RUPERT'S CUBE

MA190 GROUP PROJECT

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Contents

1 Historical Background

In the late 17th century, a wager was made between two of the most prominent scientific minds of the period; Prince Rupert of the Rhine, and the English mathematician John Wallis. The wager, as recounted by Wallis, was proposed by Rupert, who wagered that a hole could be cut through a cube, such that a cube of identical dimensions could be passed through it, without bending or stretching either cube, and without splitting the cube into pieces.

Figure 1: Prince Rupert of the Rhine. [Source: Wikimedia Commons]

Prince Rupert was a founding member of the Royal Society, where he was responsible for several innovations, many of which were military-related, such as a more powerful form of gunpowder and the Rupertinoe naval gun. His life was warfare centered from a young age, serving as a child soldier during the Eighty Years' War, a commander in the English Civil War, and as a colonial governor in Gambia.

Figure 2: John Wallis. [Source: Wikimedia Commons]

John Wallis on the other hand, had a much less disturbing background, being an English clergyman and prominent mathematician, most noted for his work on the development of infinitesimal calculus and being credited with the introduction of the symbol ∞ to represent the concept of infinity. Other interesting achievements of his include coining the term "momentum" and serving as the chief cryptographer for the British parliament between 1643 and 1689, and later for the royal court. He was a fellow member of the

Royal Society to Rupert, and was also involved in the Society from very early in its beginnings, where he and Rupert likely met.

Wallis eventually showed that the hole that Rupert had described in his wager was indeed possible, albeit with some errors that were not corrected for approximately another hundred years, losing the wager to Rupert. These errors were corrected by the renowned Dutch mathematician Pieter Nieuwland, when he found the optimal solution to the problem, using a hole of a different angle to that of Wallis' solution. Nieuwland was an extremely prominent academic in his time, sometimes being referred to as "The Dutch Isaac Newton". He was a poet, and a lector in numerous subjects at the University of Amsterdam, including mathematics and astronomy. Nieuwland sadly died before he could take credit for his improved solution to the problem, but it was published on his behalf around 32 years later by his former mentor, Jean Henri van Swinden. Because of this, the optimal solution to the problem was not known to the wider world for about 132 years after the problem was originally conceived. The solution that Nieuwland produced sometime around 1793 (the exact date is not known) was the optimal solution to the problem, and remains so to this day.

Figure 3: Pieter Nieuwland. [Source: Wikimedia Commons]

2 Introduction To The Problem

The problem itself was as follows: Given a cube, is it possible to cut a hole such that a cube of equal or greater size can be passed through it without splitting the original cube?

At first, given the intuitive geometry of the proposition it seems impossible to complete for two cubes of the same size, let alone to fit a larger cube through a smaller cube. This is, however, entirely possible.

A less difficult to imagine example of an object having this property is a cuboid. Imagine, for example, a rectangular prism, with height = 2cm, width = 10cm, and length = 20cm.

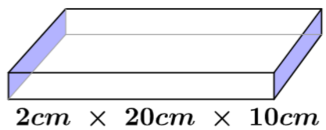

Figure 4: Rectangular prism. [Source: geogebra.org]

It is not difficult at all to imagine cutting a hole in this shape, such that an identical cuboid could be passed through it, it simply requires the shape to be oriented such that a large plane on the first is cut into, and a smaller plane on the second is passed through it. For the example prism shown above, one would simply have to orient the second prism such that the highlighted blue side is perpendicular to the vertical face of the first prism, and rotate the second prism $90°$, and it would easily pass through the first prism.

A similar technique can be employed for Prince Rupert's Cube, one simply has to orient the first cube such that the plane facing the second cube is greater than the plane it is facing on the second cube. This may seem impossible, as all the faces of a cube are identical, but not all planes of a cube are identical. If a cube is oriented such that a vertex is facing the viewer, the plane shown is hexagonal, not square. When oriented as such, then a hole can be cut in the cube such that an identical (or greater) cube could pass through it.

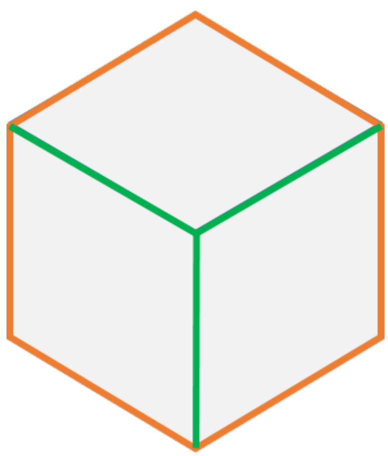

Figure 5: Hexagonal projection of a cube. [Source: momentofmaths.roots2grow.org]

3 Attempted Solutions

The first solution to this problem was proposed by John Wallis in 1693. Although his solution was accepted at the time, it was later found to be flawed, and not the optimal solution. Wallis' main error came from the fact that he assumed that the hole through the cube would be parallel to a "space diagonal" of the cube (a hypothetical line which connects two opposite vertices of a cube).

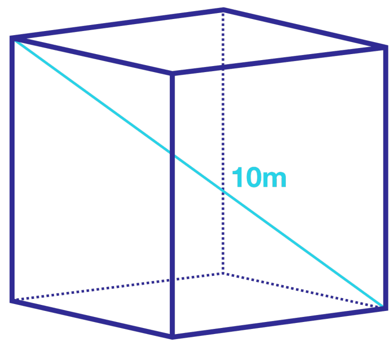

Figure 6: Diagram showing the space diagonal of a cube. [Source: brilliant.org]

The projection of the cube onto a plane perpendicular to this space diagonal is a regular hexagon. The best hole parallel to the diagonal would then be found by drawing the largest possible square on that hexagon.

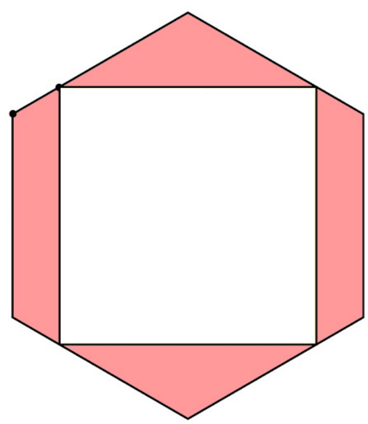

Figure 7: Diagram showing the largest possible square within a hexagon. [Source: Original Image]

If we calculate the size of the square, it shows that a cube with side length slightly larger than one is capable of passing through the hole. If we assume the cube in question to be a unit cube, i.e one with side length of 1 unit, the side length of the hole is calculated as follows:

$$\sqrt{6} - \sqrt{3} \approx 1.03527$$

This proof showed that not only could a hole be made in a cube such that a cube of identical dimensions could be passed through it, but that a hole could be made in a cube such that a cube of slightly larger dimensions could be passed through it.

This remained the commonly accepted solution to the problem for roughly 100 years, until the optimal solution was discovered by Pieter Nieuwland, a Dutch mathematician. Nieuwland found that the optimal solution could be achieved by using a hole with a different angle than the space diagonal.

4 Solution To The Problem

In a unit cube where each side is 1 unit length, and two points are placed adjacently at a distance of 3/4 from the point where those edges meet, then the distance between the points will be

$$\frac{3\sqrt{2}}{4} \approx 1.0607$$

If two more points are placed symmetrically to these on the opposite face of the cube, a square is formed within the cube through which a hole could be cut such that it would allow for a cube of side length $\frac{3\sqrt{2}}{4}$ to pass through, which is actually slightly larger than the original cube.

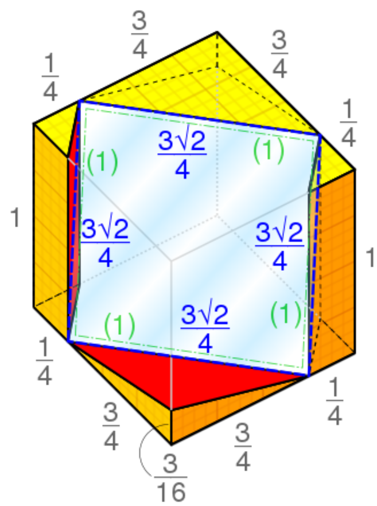

Figure 8: Diagram showing the optimal hole. [Source: Wikimedia Commons]

The remaining shell of the cube through which the hole was cut forms two trianguler prisms, and two irregular tetrahedra, which form extremely thin connections at the four vertices of the square.

A cube of side length 1.0607 or less can be passed through this hole, which includes cubes of identical dimensions (side length = 1). This is the optimal solution to the problem, further proving Rupert correct, and finding the maximum sized cube that can be passed through a unit cube.

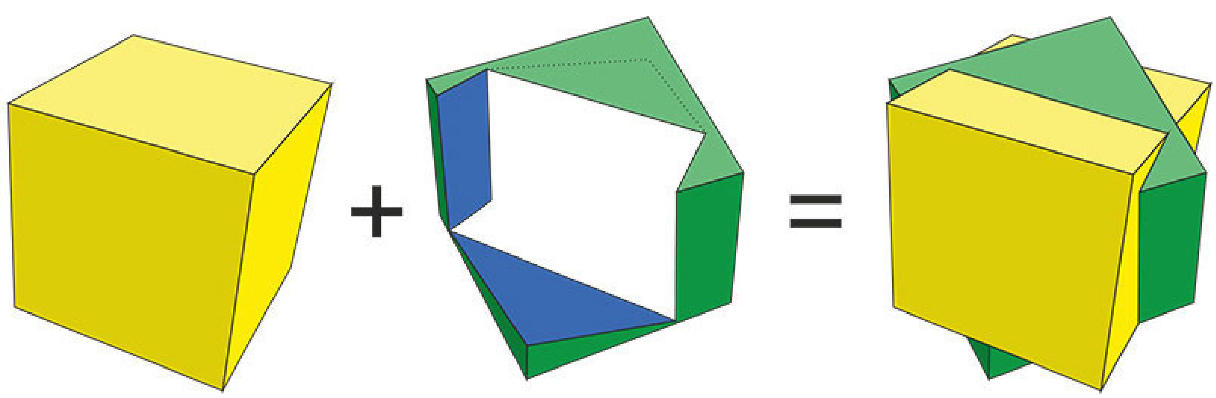

Figure 9: Diagram showing the cube passing through the optimal hole. [Source: polyhedr.com]

5 The Rupert Property

The Rupert Property is a property given to a polyhedron if it is possible for a polyhedron of the same shape and of the same size can pass through a hole in the given polyhedron. The five platonic solids (The Cube, the regular Tetrahedron, the regular Octahedron, the regular Dodecahedron and the regular Icosahedron) have all been proven to possess the Rupert property.

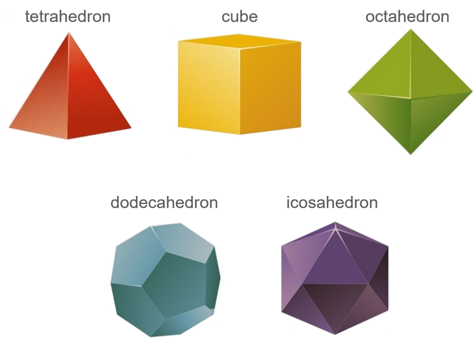

Figure 10: The platonic solids. [Source: donstewart.blogspot.com]

Of the 13 Archimedean solids (Solids that are made up of regular polygons excluding platonic solids, prisms and antiprisms) 9 of them have been shown to posses the Rupert property, these being the cuboctahedron, truncated octahedron, truncated cube, rhombicuboctahedron, icosidodecahedron, truncated cuboctahedron, truncated icosahedron, truncated dodecahedron and truncated tetrahedron respectively.

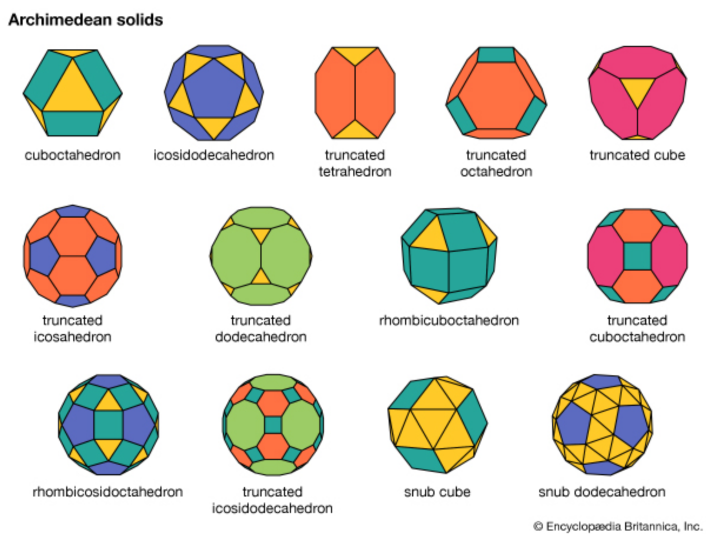

Figure 11: Archimedean solids. [Source: britannica.com]

It has been conjectured that all convex polyhedrons possess this property but as of the time of writing, it is yet to be proven. It is known however that any n dimensional cube where n is more than 2 does possess the Rupert property.

6 References

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