CS4423: Networks

Week 8, Part 1: Introduction to Random Networks

Dr Niall Madden

School of Maths, University of Galway

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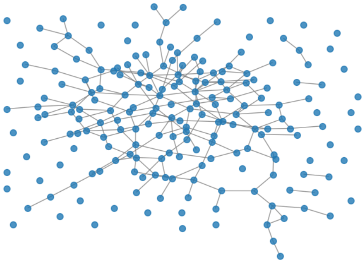


Class Test!

Class Test 2pm tomorrow!

Details:

- LENS reports: email Niall today!
- Locations: see announcement!
- Content:
  - Similar to Problem Set 2
  - Nothing from this week.
  - No `networkx`
  - Focus on skills, rather than theory.
- Bring a pen. And maybe a calculator (?).
- If you miss the test, for any reason, your grade will be based on the assignments (20%) and the final exam (80%).

Outline

Today's notes are split between these slides, and a Jupyter Notebook.

1. Random Models of Networks
2. Erdö-Rényi Random Graph Models
   - Some examples
3. Random samples
4. The two Erdös-Rényi Models
   - Model A: $G_{ER}(n, m)$
   - Model B: $G_{ER}(n, p)$

Slides are at:

https://www.niallmadden.ie/2425-CS4423

Random Models of Networks

One of the remaining "big" ideas for us to study in CS4423 is that of **Random Networks**. In a sense, we are not so interested in their randomness. It is more like we decide on the general structure of networks, but then choose a particular example by tossing a coin, or rolling dice.

What we are interested in:

- The **statistical properties** of very large networks, such as average degree, the number of 3-cycles, or the size of component.
- How well our random networks share these properties.

Erdö-Rényi Random Graph Models

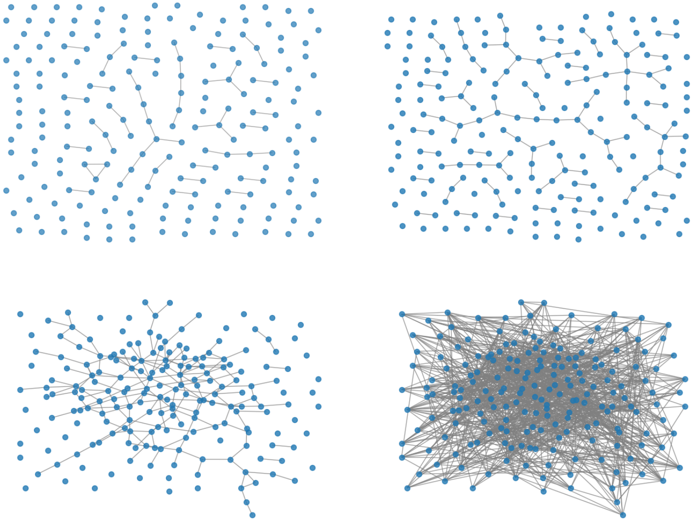
A **Random Graph**[1] is a *mathematical model* of a family of networks, where certain parameters (like the number of nodes and edges) have fixed values, but other aspects (like the actual edges) are randomly assigned.

The simplest example of a random graph is in fact a network with fixed numbers n of nodes and m of edges, randomly placed between the vertices.

Although a random graph is not a specific object, many of its properties can be described precisely in the form of **expected values** or **probability distributions**.

[1] https://en.wikipedia.org/wiki/Random_graph

Erdö-Rényi Random Graph Models — Some examples



Random samples

Suppose our network $G = (X, E)$ has $|X| = n$ nodes. Then we know the most number of edges t can have is:

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}.$$

- Our goal is to randomly select edges on the vertex set X. That is, pick at random elements from the set $\binom{X}{2}$ of pairs of nodes.
- So we need a procedure for selecting m from N objects randomly, in such a way that each of the $\binom{N}{m}$ subsets of the N objects is an equally likely outcome.
- We first discuss sampling m values in the range $\{0, 1, \ldots, N-1\}$.

Random samples

1. Suppose we choose a natural number N, and real number $p \in [0, 1]$
2. Then iterate over each element of the set $\{0, 1, \ldots, N-1\}$.
3. For each, we pick a random number $x \in [0, 1)]$.
4. If $x < p$, we keep that number. Otherwise remove it from the set.

When we are done, how many elements do we expect in the set if $p = m/N$ for some chosen m?
And what is the likelihood if there being, say k elements in the set?

Random samples

We are creating random samples. The size of each is a random number, k.

Claim: Expected value: $E[k] = Np = m$.

Proof: This is a **binomial distribution**[2]

- The probability of a specific subset of size k to be chosen is $p^k(1-p)^{N-k}$.
- There are $\binom{N}{k}$ subsets of size k. So the probabilty $P(k)$ of the sample to have size k is $P(k) = \binom{N}{k}p^k(1-p)^{N-k}$.

We use the following facts

(i) $j\binom{N}{j}p^j = Np\binom{N-1}{j-1}p^{j-1}$,

(ii) $(1-p)^{N-j} = (1-p)^{(N-1)-(j-1)}$,

(iii) $(p+(1-p))^r = 1$ for all r.

[2] https://en.wikipedia.org/wiki/Binomial_distribution

Random samples

Expected value:

$$E[k] = \underbrace{\sum_{j=0}^{N} jP(j)}_{\text{weighted average of } j} = \sum_{j=0}^{N} j \underbrace{\binom{N}{j} p^j (1-p)^{N-j}}_{\text{Formula for } P(j)}$$

$$= \underbrace{Np \sum_{l=0}^{N-1} \binom{N-1}{l} p^l (1-p)^{(N-1)-l}}_{\text{From (i),(ii),(ii)}} = Np, \quad (1)$$

substituting $l = k - 1$,

Random samples

Next week, we'll look a some computational examples, as well as an algorithm for choosing exactly m numbers from a set of N.

For now, we'll just assume it can be done...

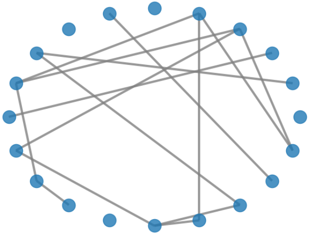
Uniformly selected edges

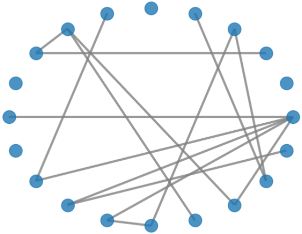
ER Model $G_{ER}(n, m)$: Uniform Random Graphs

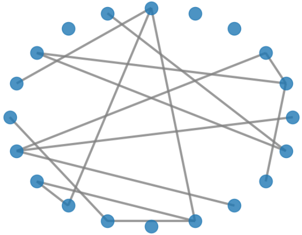
Let $n \geq 1$, let $N = \binom{n}{2}$ and let $0 \leq m \leq N$.
The model $G_{ER}(n, m)$ consists of the ensemble of graphs G on the n nodes $X = \{0, 1, \ldots, n-1\}$, and m randomly selected edges, chosen uniformly from the $N = \binom{n}{2}$ possible edges.

Equivalently, one can choose uniformly at random one network in the **set** $\mathcal{G}(n, m)$ of *all* networks on a given set of n nodes with *exactly* m edges.

The two Erdös-Rényi Models   Model A: $G_{ER}(n, m)$







Randomly selected edges

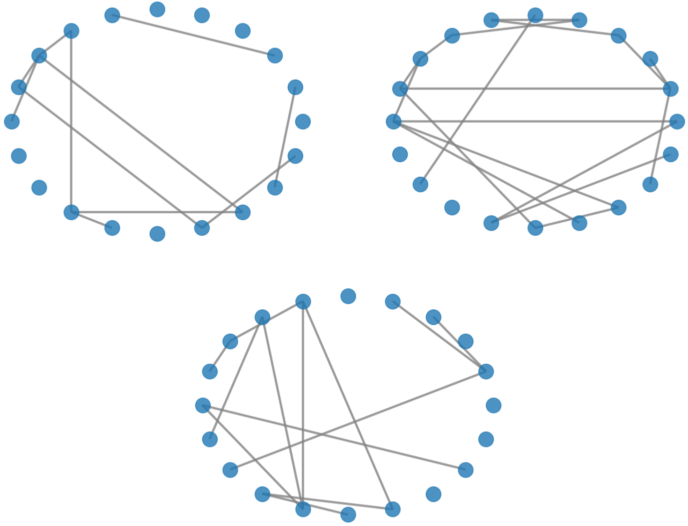
ER Model $G_{ER}(n, p)$: Random Edges

Let $n \geq 1$, let $N = \binom{n}{2}$ and let $0 \leq p \leq 1$.
The model $G_{ER}(n, p)$ consists of the ensemble of graphs G on the n nodes $X = \{0, 1, \ldots, n-1\}$, with each of the possible $N = \binom{n}{2}$ edges chosen with probability p.

The probability $P(G)$ of a particular graph $G = (X, E)$ with $X = \{0, 1, \ldots, n-1\}$ and $m = |E|$ edges in the $G_{ER}(n, p)$ model is

$$P(G) = p^m (1 - p)^{N-m}.$$

The two Erdös-Rényi Models — Model B: $G_{ER}(n, p)$



Of the two models, $G_{ER}(n, p)$ is the more studied. They are many similarities, but do differ. For example:

1. $G_{ER}(n, m)$ will have m edges with probability 1.
2. A graph in $G_{ER}(n, p)$ with have m edges with probability $\binom{N}{m} p^m (p-1)^{N-m}$.