Annotated slides from Wednesday

CS4423: Networks

Lecture 7: Permutations and Bipartite Networks

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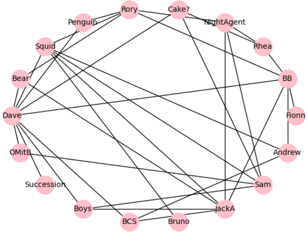
Week 4, Lecture 1 (Wed, 5 Feb 2025)



These slides are by Niall Madden. Elements are based on "A First Course in Network Theory" by Estrada and Knight. Also AC's notes...

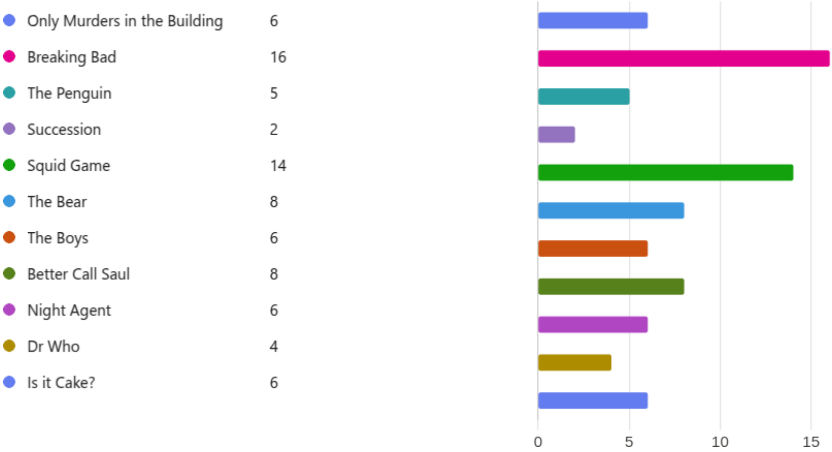
Outline

For further reading, see Section 2.4 of A First Course in Network Theory (Knight).

Slides are at:
https://www.niallmadden.ie/2425-CS4423

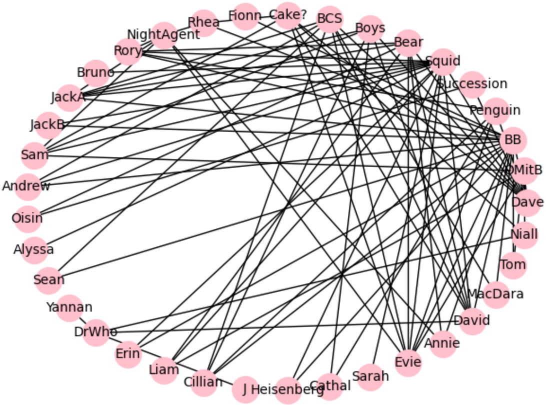
Thanks for completing the survey!

Here is some of the data we collected:



Thanks for completing the survey!

Here is what it looks like as a graph:



Its order is 37, and size is 81; we'll return to this later...

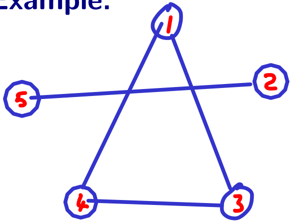
Graph Connectivity

- A graph/network is **connected** if there is a path between every pair of nodes.
- If the graph is *not* connected, we say it is **disconnected**.
- We now know how to check if a graph is connected by looking at powers of its adjacency matrix. However, that is not very practical for large networks.
- However, we can determine if a graph is connected, but just looking at the adjacency matrix, providing we have ordered the nodes properly.

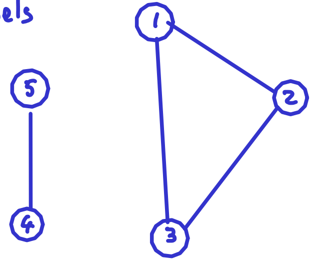
Permutation matrices

We know that the structure of a network is not change~~s~~d by relabelling its nodes. Sometimes, it is is useful to relabel them in order to expose certain properties, such as connectivity.

Example:



Swap labels of 2 & 4.



Since we think of the nodes as all being numbered from 1 to n, this is the same as **permuting** the numbers of some subset of the nodes.

Permutation matrices

When working with the adjacency matrix of a graph, such a permutation is expressed in terms of a **permutation matrix**, P: this is a $0-1$ matrix (a.k.a. a "Boolean" or "binary" matrix), where there is a single 1 om every row and column.

If the nodes of a graph G (with adjacency matrix A) are listed as entries in a vector, q, then

- Pq is a permutation of the nodes, and
- $\boxed{PAP^T}$ is the adjacency matrix of the graph with that node permutation applied.

Notes:

- In many examples we'll see, P is symmetric. – but that is just for symplicity. In general $P \neq P^T$
- But $P^TAP = PAP^T$
- Also $P^T = P^{-1}$ so $PAP^T = PAP^{-1}$

Permutation matrices are important when studying graph connectivity because...

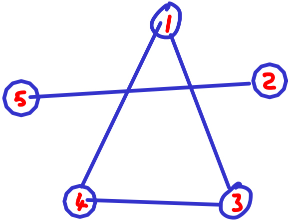
FACT!

A graph with adjacency matrix A is **disconnected** if and only if there is a permutation matrix P such that

$$\cancel{A = P\begin{pmatrix} X & O \\ O^T & Y \end{pmatrix} P^T} \qquad PAP^T = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$$

where O represents the zero matrix with the same number of rows as X and the same number of columns as Y.

Example: From Earlier: To swap labels of nodes 2 & 4:



$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

And $$PAP^T = \left(\begin{array}{ccc|cc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

If a network is not connected, then we can divide it into **components** which *are* connected.

The number of connected components is the number of blocks in the permuted adjacency matrix:

Eg : a graph with $\underline{\underline{3}}$ connected componets

$$
\left(\begin{array}{cc|ccc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

"Block Diagonal"!

Bipartite Graphs (again)

One reason we did the survey is that the resulting data set is a good example of a **bipartite** graph: nodes represent either people or programmes that they watch, with an edge between a person and a programme that they watch.

So the graph must be bipartite.

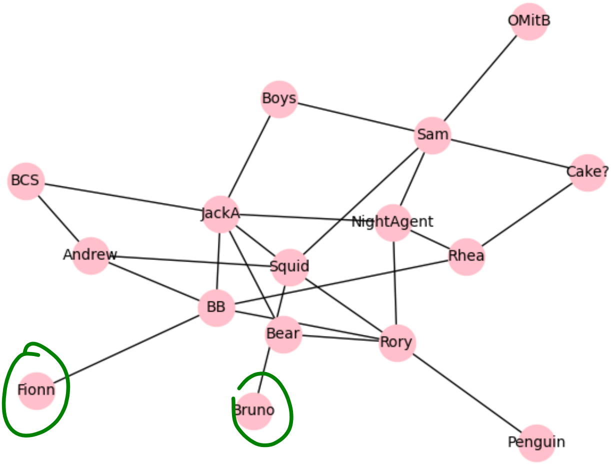
Such a graph is called an **affiliation** network;

In social sciences:

- People are called "actors"
- Programms would be "foci".

Bipartite Graphs (again)

Here is a **subgraph** of our survey, of order 16 and size 24, based on 7 randomly chosen people:



Bipartite Graphs (again)

This is the adjacency matrix: This is a 16 x 16 matrix.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

The row sums are the degree of the node. Eg if a person, it is number of progs they watch. Etc.

Bipartite Graphs (again)

That version of the adjacency matrix is not very insightful. But ordering the nodes so that people are listed first we get the matrix:

People

Programmes

$$A = \left(\begin{array}{ccccccc|ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Bipartite Graphs (again)

Person 1 & 4 dont watch the same shows.

Let's consider $B = A^2$:

Person 5 & 3 have 4 shows in common.

$$B = \left(\begin{array}{ccccccc|ccccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 5 & 1 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 4 & 1 & 6 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 1 & 3 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 3 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 3 & 1 & 1 & 3 & 2 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 2 & 1 & 3 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 1 & 1 & 5 & 2 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}\right)$$

Finished here Wed at 10.