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Lect 3: Dot Product & hyperplanes

def^n: Let $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

The norm (or length) of v denoted $\|v\|$ is defined to be

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Properties of the Dot Product

i) $u \cdot v = v \cdot u \quad \forall u, v \in \mathbb{R}^n$ "$\forall$ = for every"

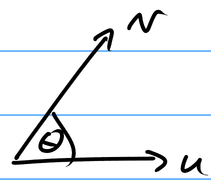
ii) $u \cdot (v + w) = u \cdot v + u \cdot w \quad \forall u, v, w \in \mathbb{R}^n$

iii) $(ru) \cdot v = u \cdot (rv) = r(u \cdot v) \quad \forall u, v \in \mathbb{R}^n, \forall r \in \mathbb{R}$

iv) In $\mathbb{R}^2$ we will prove that

$$u \cdot v = \|u\| \|v\| \cos\theta$$

where θ is the angle between u & v.



As in higher dimensions any 2 vectors define a 2-dimensional plane; we also have the same notion of angle θ.

Corollary: $u \cdot v = 0 \iff \cos\theta = 0$

$$\Rightarrow \theta = \pi/2$$

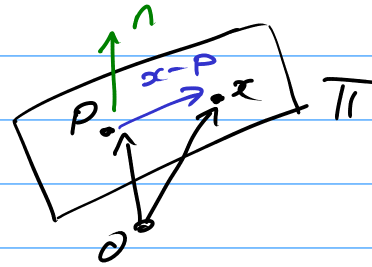
So u & v are perpendicular or orthogonal.

def^n: The hyperplane in $\mathbb{R}^n$ with normal $n = (n_1, n_2, \dots, n_n)$ through the origin $O = (0, 0, \dots, 0)$ is the set of vectors (or points) $x = (x_1, x_2, \dots, x_n)$ that are perpendicular to n (orthogonal)

So $n \cdot x = n_1 x_1 + n_2 x_2 + \dots + n_n x_n = 0$

When $n=3$ we just call it the plane.. instead of hyperplane.

Ex If the plane goes through some other point, say $P=(p_1,p_2,p_3)$ & again has normal $n=(n_1,n_2,n_3)\neq 0$



Then the equation of the plane is

$$\pi : \{(x_1,x_2,x_3)=x \mid (x-P)\cdot n = 0\}$$

$x-P \perp^r$ to n.

So the eqn is

$$x\cdot n - P\cdot n = 0$$

ie $\quad x\cdot n = P\cdot n := d \in \mathbb{R}$

ie $\quad \boxed{n_1x_1 + n_2x_2 + n_3x_3 = d}$

Ex Find the eqn of the plane in $\mathbb{R}^3$ with normal $n=(2,3,1)$ passing through the point $P=(1,1,2)$.

By the above, if $x=(x_1,x_2,x_3)$ lies on the plane then $n\cdot x = n\cdot P \Rightarrow$

$$(2,3,1)\cdot(x_1,x_2,x_3) = (2,3,1)\cdot(1,1,2)$$

$\Rightarrow \quad 2x_1+3x_2+x_3 = 2+3+2$

ie $\quad 2x_1+3x_2+x_3 = 7$

Check that $P=(1,1,2)$ lies on this

Ans: $\quad 2(1)+3(1)+2 \overset{??}{=} 7$

$\quad 2+3+2 = 7$ ✓ So yes.

It is intuitively clear that there is a unique plane π through 3 points (in general position) in $\mathbb{R}^3$. How to find the eqn of π.

Ex: Find the eqn of the plane Π in $\mathbb{R}^3$ containing $(1,1,2)$, $(-3,4,6)$ and $(0,5,7)$.

The eqn is of the form
$$n_1 x_1 + n_2 x_2 + n_3 x_3 = d$$

$\Rightarrow$ $n_1 + n_2 + 2n_3 = d_1$ ① $(1,1,2) \in \Pi$

$-3n_1 + 4n_2 + 6n_3 = d_2$ ② $(-3,4,6) \in \Pi$

$5n_2 + 7n_3 = d_3$ ③

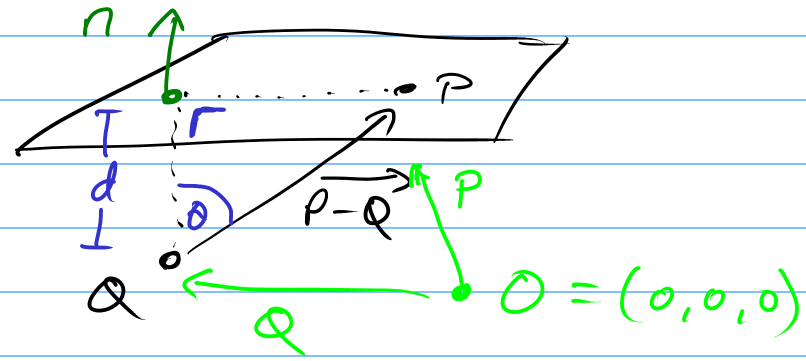
So we need to solve 3 linear eqns in the 3 unknowns n_1, n_2 & n_3 (where d_1, d_2, d_3 are given real numbers)

We expect that this is just enough to yield a unique sol^n

eg last year we saw that generally 2 eqns in 2 unknowns have a unique sol^n, namely the intersection point of the 2 lines in $\mathbb{R}^2$ with the given eqn.

Geometrically (generally) 3 planes in $\mathbb{R}^3$ intersect in just one point (2 intersect in a line & this line meets the 3rd plane in a unique point)

Find the distance d from a point Q to the plane Π with normal n & containing P. We assume $Q \notin \Pi$



$d = \|P-Q\| \cos\theta$

Also $\overrightarrow{(P-Q)} \cdot \vec{n} = \|P-Q\| \, \|n\| \cos\theta$
$= d\,\|n\|$

Thus $d = \dfrac{|(P-Q)\cdot n|}{\|n\|}$

- Solving Systems of Linear Equations and Echelon Form

Some systems are easily solved eg

$$\left.\begin{array}{rl} x_1 + 2x_2 + x_3 = 16 & \text{(i)} \\ x_2 + 2x_3 = 10 & \text{(ii)} \\ 3x_3 = 9 & \text{(iii)} \end{array}\right\} \text{system}$$

This is because of the "upper diagonal" shape of the system of eqns,
ie. eqn (ii) involves one less unknown than eqn (i) and eqn (iii) involves one less eqn than eqn (ii)

Sol^n: A solution to the above system of linear eqns
linear (i.e. more than one eqn & the unknowns x_1, x_2, x_3 can only be multiplied by reals and then added or subtracted)
is just a vector (c_1, c_2, c_3) so that if $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$ then all three eqns (i), (ii) & (iii) are satisfied.

In this case,
(iii) $\Rightarrow$ $x_3 = 3$. Back substituting into (ii)
(ii) $\Rightarrow$ $x_2 + 2(3) = 10$ $\Rightarrow$ $x_2 = 4$.
& finally back substituting into (i)
i) $\Rightarrow$ $x_1 + 2(4) + 3 = 16$ $\Rightarrow$ $x_1 = 5$

So in this case we have a unique solution $x_1 = 5$, $x_2 = 4$ & $x_3 = 3$ or simply $(5, 4, 3)$.

Aside: We will also use the definition of matrix multiplication to write a system of linear eqns as one matrix eqn.

Above let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ 3x3 rows

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1}$ & $b = \begin{pmatrix} 16 \\ 10 \\ 9 \end{pmatrix}_{3\times 1}$

Then our system can be written as

$$Ax = b$$

$(3\times 3)(3\times 1) = (3\times 1)$ sizes

Idea: Change a given system of linear eqns. into a system that has a "simple" form as above and that has the same solutions

ie to a so called "equivalent system".

Question: What can we do to a system of linear eqns without changing the solutions?

Elementary Operations:

Ans: (1) Interchange 2 equations

(2) Multiply an eqn by a nonzero real number

(3) Replace an equation by itself plus a multiple (non-zero) of some other eqn in the system.

Weds (25) When we write the system of equations in 'Extended Matrix Notation'

ie.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad (i) \\ a_{21}x_2 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad (ii) \\ \vdots \\ a_{m1}x_n + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \quad (n) \end{array}\right\}$$

Rewrite as $Ax = b$
or simply encode this info in the extended (or augmented) matrix

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$$

so that equation
(i) corresponds to Row 1 of $(A|b)$
(ii) corresponds to Row 2 of $(A|b)$
⋮
(n) " Row n "

Letting R_i denote Row i of $(A|b)$

Then the Elementary (Row) Operations are written as:

1) Interchange R_i & R_j

2) Replace R_i by kR_i, $k \neq 0$, $k \in \mathbb{R}$

3) Replace R_i by $R_i + kR_j$, $j \neq i$, $k \in \mathbb{R}$

Ex Consider the system of 3 linear eqns in 3 unknowns x_1, x_2 & x_3.

$$\left.\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 100 \\ x_1 + x_2 + 4x_3 &= 70 \\ 20x_1 + 10x_2 + 10x_3 &= 500 \end{aligned}\right\} \leftrightarrow \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

Instead, keep track in the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right) := (A|b)$$

Now use the Elementary Row Operations so that $(A|b)$ is in so called "echelon form" ie an "upper triangular" form like the first "simple" eqns we solved last lecture.

The idea is to use the first nonzero entry in a row (pivot) to place zeros in the column entries below.

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 2 & 3 & 2 & 100 \\ 20 & 10 & 10 & 500 \end{array}\right)$$

pivot

$$\xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 20R_1}]{R_1 \text{ fixed}} \begin{pmatrix} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & -10 & -70 & -900 \end{pmatrix}$$

pivot (fluke that this is 1)

$$\xrightarrow[\substack{R_2 \text{ fix (pivot)} \\ R_3 \to R_3 + 10R_2}]{R_1 \text{ fix}} \begin{pmatrix} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{pmatrix}$$

So Row 3 says $-130x_3 = -1300$

$\Rightarrow \quad x_3 = 10$

Row 2 says: $x_2 - 6x_3 = -40$

$\Rightarrow \quad x_2 - 6(10) = -40 \quad \Rightarrow x_2 = 20$

& Row 1 says: $x_1 + x_2 + 4x_3 = 70$

$\Rightarrow \quad x_1 + 20 + 4(10) = 70$

$\Rightarrow \quad x_1 = 10$

defn: We say that a system of linear eqns is

- consistent if it has a solution ie either a unique soln or infinitely many
- inconsistent otherwise ie if it has no soln.

Ex: The above system has a unique solution, so is consistent.

Ex Find the sol^s ~~of~~, if any, of

$$\left.\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}\right\}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - \frac{5}{2}R_1]{R_1, R_2 \text{ fixed}} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -1\frac{1}{2} \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + \frac{1}{2}R_2]{} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 2\frac{1}{2} \end{pmatrix}$$

Row 3 says $0x_1 + 0x_2 + 0x_3 = 2\frac{1}{2}$

impossible : INCONSISTENT SYSTEM.

So there is no solution to the system.

Lect 6: Echelon Form & Reduced Echelon Form

Recall in solving the system of eqns with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right)$$

we used elementary row operations to put it in echelon form

$$\xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right)$$

& used back-substitution to get $x_1 = 10,\ x_2 = 20\ \&\ x_3 = 10$

Alternatively, we could have continued & placed as many zeros as possible above the pivots in each column as follows to put the matrix (augmented) into Reduced Echelon Form

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right) \xrightarrow[R_3 \to -R_3/130]{} \begin{pmatrix} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & 1 & 10 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 6R_3]{R_1 \to R_1 - 4R_3} \begin{pmatrix} 1 & 1 & 0 & 30 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 10 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 10 \end{pmatrix}}_{\text{reduced echelon form}}$$

$$\Longleftrightarrow \quad x_1 = 10, \quad x_2 = 20, \quad x_3 = 10$$

The reduced echelon form is useful when looking for all solutions ie the "general solⁿ" of a system with infinitely many solutions

Ex: Find the general solⁿ of the system whose augmented matrix has been reduced to

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

Nice surprise! The augmented matrix is already in echelon form (the 1st nonzero element (pivot) in each row is to the right of the row above it). Now put it in reduced echelon form (use pivots to put zeros above them).

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_3,\ R_3 \text{ fixed}]{R_1 \to R_1 + 2R_3} \begin{pmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

So
$$\begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

The columns with a pivot (a 1st nonzero in a row) are columns 1, 3 & col. 5

$\longleftrightarrow$ non-free variables x_1, x_3 & x_5

The remaining variables x_2 & x_4 are called free variables as they are free to be any real numbers & the non-free variables are determined in terms of the free ones as follows:

$$\begin{aligned} x_1 &= -6x_2 - 3x_4 \\ x_3 &= 4x_4 + 5 \\ (x_5 &= 7) \end{aligned}$$

$\Rightarrow$ x_2 is free ie $x_2 = s \in \mathbb{R}$

$\Rightarrow$ x_4 is free, say $x_4 = t \in \mathbb{R}$

So the "solution space" is the set of all solutions i.e.

$$S = \{(-6s - 3t, s, 4t+5, t, 7) \in \mathbb{R}^5 : s, t \in \mathbb{R}\}$$

Exercise: Take special values for s & t eg $s=3$, $t=2$ & verify that $(-24, 3, 13, 2, 7)$ does indeed satisfy the original eqns.

or let $s=0$, $t=0$ so that $(0,0,5,0,7)$ is a solution of $Ax=b$.

Note: The set of all solutions S can also be written as

$$S = \{(-6s-3t, s, 4t, t, 0) + (0,0,5,0,7) \in \mathbb{R}^5 : s,t \in \mathbb{R}\}$$

and (check!)

$S' = \{(-6s-3t, s, 4t, t, 0) \in \mathbb{R}^5 : s,t \in \mathbb{R}\}$

is the set of all solutions of the system $Ax = \begin{pmatrix}0\\0\\0\end{pmatrix}$; $A = \begin{pmatrix}1 & 6 & 2 & -5 & -2\\ 0 & 0 & 2 & -8 & -1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$, $x = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$

called the corresponding HOMOGENEOUS system & $P = (0,0,5,0,7)$ is one "particular" sol^n of $Ax = b$ (check!)

This is a general fact for systems of (consistent) linear eqns ie the sol^{n}s of $Ax=b$ are all of the form $x' + P$ where x' is a sol^n of $Ax = 0$ and P is one "particular" soln of $Ax = b$.

Ex: Solve the homogeneous system of eqns:

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 3x_2 + 4x_3 - 3x_4 &= 0 \\ 3x_1 - 5x_2 + 5x_3 - 4x_4 &= 0 \\ -x_1 + x_2 - 3x_3 + 2x_4 &= 0. \end{aligned}$$

(47): Ans: The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ 3 & -5 & 5 & -4 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array}\right) \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + R_1}]{\substack{R_1 \text{ fixed} \\ R_2 \to R_2 - 2R_1}} \begin{pmatrix} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2}]{R_2 \text{ fix}} \begin{pmatrix} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(echelon form)}$$

$R_1 \to R_1 + 2R_2$
R_2 fixed $\longrightarrow$ $\begin{pmatrix} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (reduced echelon form)

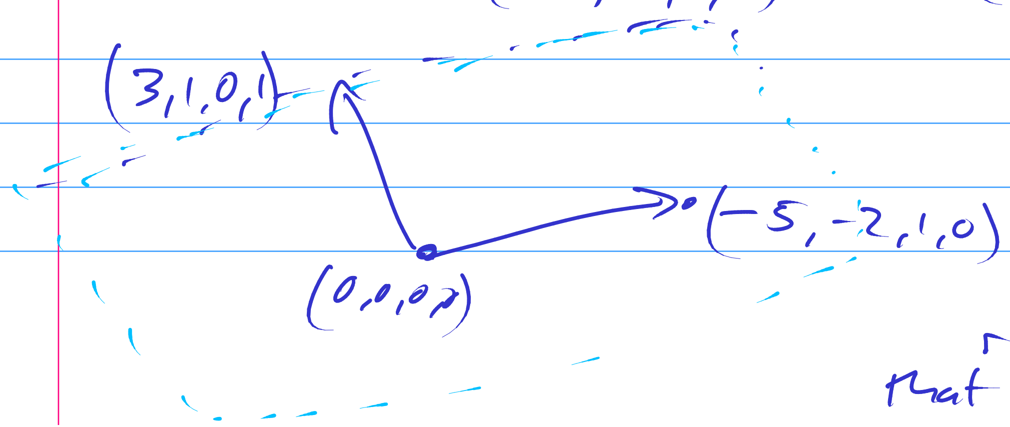
So the columns with a (nonzero) pivot are Col. 1 & Col. 2 so x_1 & x_2 are non-free (or basic) variables and the others ie x_3 & x_4 are free variables ie they can be any real numbers r and s.
So the general sol^n S (or the set of all solutions S) is

$$\begin{aligned} x_1 &= -5x_3 + 3x_4 \\ x_2 &= -2x_3 + x_4 \\ x_3 &= r \\ x_4 &= s \end{aligned}$$

ie. $S = \{(-5r+3s, -2r+s, r, s) \in \mathbb{R}^4 : r, s \in \mathbb{R}\}$

$= \{r(-5,-2,1,0) + s(3,1,0,1) \in \mathbb{R}^4 : r, s \in \mathbb{R}\}$

Geometrically S is a 2-dimensional plane in $\mathbb{R}^4$ through the origin $(0,0,0,0)$ because if $x \in S$ then $x = (-5r+3s, -2r+s, r, s)$
$= r(-5,2,1,0) + s(3,1,0,1)$ & $r,s \in \mathbb{R}$



Note: letting $r = s = 0$ we see that $(0,0,0,0) \in S$

§ Linear Independence of vectors in $\mathbb{R}^n$

We have seen examples of systems of linear eqns which contain redundant equations in the sense that some of the equations may be obtained from others, ie in the sense that some of the equations can be written as linear combinations of (the) others.

Equivalently given a collection of vectors in $\mathbb{R}^n$ can we write some of them as a linear combination of the others?

If yes, we say the collection is linearly dependent.
And if not, we say the collection is linearly independent.

Ex: Is the collection
$$v_1 = (1,2,1),\ v_2 = (3,-1,2),\ v_3 = (9,4,7)$$
linearly independent?

Ans: No the vectors are linearly dependent as $(9,4,7) = 3(1,2,1) + 2(3,-1,2)$

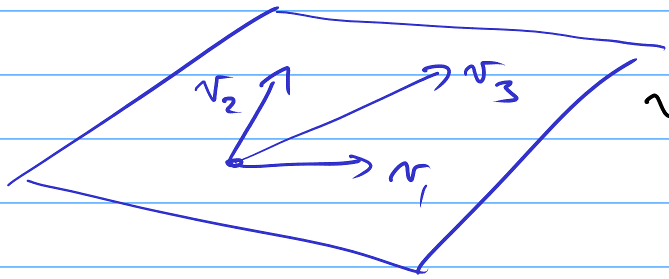
$$\iff 1.(9,4,7) - 3(1,2,1) - 2(3,-1,2) = (0,0,0)$$

ie there exists $x_1, x_2, x_3 \in \mathbb{R}$ not all zero s.t. $x_1(9,4,7) + x_2(1,2,1) + x_3(3,-1,2) = (0,0,0)$

(Here $x_1 = 1$, $x_2 = -3$ & $x_3 = -2$)

L8 Geometrically: $v_3 = (9,4,7)$ lies in the plane ~~Π~~

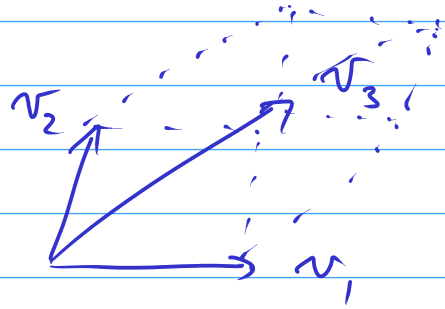
$$\Pi := \{v \in \mathbb{R}^3 : v = sv_1 + tv_2 ; s,t \in \mathbb{R}\}$$



$v_3 = 3v_1 + 2v_2$

If v_1, v_2 and v_3 were linearly independent they would not lie in a plane & we would need 3 dimensions to draw them.

eg



They form the sides of a box

Formal def^n: A collection of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if (the vector eq^n)

$$x_1v_1 + x_2v_2 + \dots + x_pv_p = 0 \quad = (0,0,\dots,0) \in \mathbb{R}^n$$

has only the trivial solution

$$x_1 = x_2 = \dots = x_p = 0 \quad \in \mathbb{R}$$

and the collection of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ ~~is~~ is said to be linearly dependent if there exist real numbers $c_1, c_2, \dots, c_p$ not all zero s.t. $c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$

(ie the vector eq^n $x_1v_1 + x_2v_2 + \dots + x_pv_p = 0$ has a nontrivial solution)

Ex Let $v_1 = (1,2,3)$, $v_2 = (4,5,6)$, & $v_3 = (2,1,0)$. Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

ie Can we find x_1, x_2 & x_3 not all zero s.t. $x_1(1,2,3) + x_2(4,5,6) + x_3(2,1,0) = (0,0,0)$.

ie
$$\left.\begin{array}{rl} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + 5x_2 + 1x_3 &= 0 \\ 3x_1 + 6x_2 + 0x_3 &= 0 \end{array}\right\}$$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

So x_1 & x_2 are non-free (or basic) variables & x_3 is free (ie free to take any real number t as its value). and then x_1 & x_2 are determined by

$$-3x_2 - 3x_3 = 0 \implies x_2 = -x_3 = -t$$

& $x_1 + 4x_2 + 2x_3 = 0$

$$\implies x_1 = -4x_2 - 2x_3 = -4(-t) - 2t$$

All sol^{n}s are $\{(2t, -t, t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$

$$= \{t(2,-1,1) \in \mathbb{R}^3 : t \in \mathbb{R}\}$$

In particular we can choose $t \neq 0$, eg $t = 1$ to get

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1$$

So $2v_1 - v_2 + v_3 = 0$

so v_1, v_2 & v_3 are not linearly independent ie they are linearly dependent as we have found x_1, x_2 & x_3 not all zero with $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

Alternatively $v_3 = -2v_1 + v_2$ ie v_3 lies in the span of v_1 & v_2 which we now define.

def^n: If $v_1, v_2, \ldots, v_p$ are vectors in $\mathbb{R}^n$ the set of all linear combinations of $v_1, \ldots, v_p$ ie $\{v \in \mathbb{R}^n : v = c_1 v_1 + \ldots + c_p v_p : c_1, c_2, \ldots, c_p \in \mathbb{R}\}$ is called the subset of $\mathbb{R}^n$ spanned by $v_1, \ldots, v_p$ (or generated by) and is denoted by

$$\text{Span}\{v_1, \ldots, v_p\} \quad \text{or} \quad \langle v_1, \ldots, v_p \rangle$$

ie a vector $w \in \mathbb{R}^n$ is in the span of $v_1, \ldots, v_p$ if there exists a sol^n of the vector eq^n

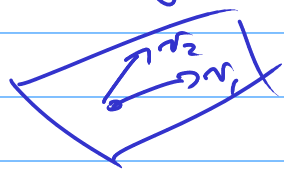
$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = w.$$

Geometrically: The span of one vector v

$\text{span}\{v\} = \{tv \mid t \in \mathbb{R}\}$

ie is the line through the origin in the direction v.

The span of 2 vectors v_1 and v_2 (not multiples of each other) is the plane through the origin that they lie in.



Ex: Prove that the vectors $v_1 = (1,-1,0)$, $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$ are linearly independent.

ie show that the only sol^n of the vector eq^n

$$x_1(1,-1,0) + x_2(1,0,2) + x_3(0,1,1) = (0,0,0)$$

is the trivial sol^n $x_1 = 0, x_2 = 0, x_3 = 0$.

ie
$$\begin{aligned} x_1 + x_2 &= 0 \\ -x_1 + x_3 &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned}$$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow[R_3 \times -1]{} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$R_1 \to R_1 + R_3$
$R_2 \to R_2 - R_3$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = 0$$

Geometrically: The vectors v_1, v_2 & v_3 don't lie in a plane but point into 3 different "dimensions".