$$L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longrightarrow L(v)$$

Recap: L linear if

- $L(u+v) = L(u) + L(v) \quad \forall u,v \in \mathbb{R}^n$
- $L(rv) = rL(v) \quad \forall v \in \mathbb{R}^n, \ \forall r \in \mathbb{R}$

Then $L \longleftrightarrow$ matrix A where $A = \begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \cdots & L(e_n) \\ | & | & & | \end{pmatrix}$

and $L(v) \Leftrightarrow Av$

is a $m \times n$ matrix (with respect to standard bases in $\mathbb{R}^n$ & $\mathbb{R}^m$)

Observe:

1) L maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$

Because
$$\begin{aligned} L(0+0) &= L(0) \\ \shortparallel \quad & \\ L(0) + L(0) &= L(0) \\ \Rightarrow \quad L(0) &= 0 \end{aligned}$$

Alternatively using the matrix $A \Leftrightarrow L$

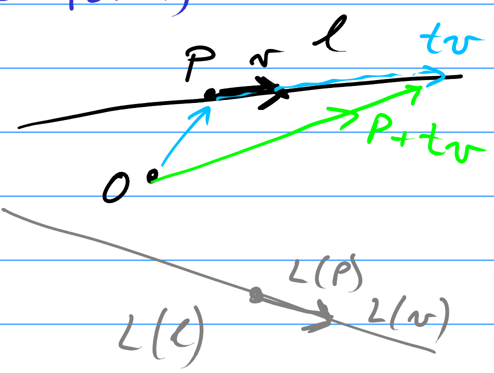
$L(0)$ is obtained as $A \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\in \mathbb{R}^n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m$

* ii) A linear transformation $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ sends lines to lines because given a line ℓ in $\mathbb{R}^n$ (in parametric form)

$$\ell: P + tv, \quad t \in \mathbb{R}$$

then

$$\begin{aligned} L(\ell) &= L(P+tv) \\ &= L(P) + L(tv) \\ &= L(P) + tL(v) \end{aligned}$$



This is the parametric form of the line (in $\mathbb{R}^m$) through the point $L(p)$ in the direction $L(v)$

?? (and $L(v) \neq 0$ if $v \neq 0$)

Examples of Linear Transformations

<u>Ex</u>: Fix a vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$

Define $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as follows:

$$L : v \longrightarrow n \times v$$

($\times$ = vector cross product)

Exercise: Check that L is linear ie

$$n \times (v + w) = n \times v + n \times w$$

$$\& \quad n \times (kv) = k(n \times v) \qquad k \in \mathbb{R}$$

Aside: Let $u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

Then $u \times v$

"u cross v" $= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$

Then $u \times v$ is $\perp^r$ to u

& is $\perp^r$ to v (assuming $u \nparallel v$)

Check: $(u \times v) \cdot v$ [should be 0 if they are $\perp^r$]

$$(u \times v) \cdot v = u_2v_3v_1 - u_3v_2v_1 + u_3v_1v_2 - u_1v_3v_2 + u_1v_2v_3 - u_2v_1v_3$$

$$= 0$$

[Exercise: check that $(u \times v) \cdot u = 0$ too]

Mnemonic for $u \times v$

$u = u_1e_1 + u_2e_2 + u_3e_3$

$e_1 = (1,0,0)$ etc

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

determinant (see later) is $e_1(u_2v_3 - u_3v_2) - e_2(u_1v_3 - u_3v_1)$
$+ e_3(u_1v_2 - u_2v_1)$

$= (u_2v_3 - u_3v_2,\ u_1v_3 - u_3v_1,\ u_1v_2 - u_2v_1)$

End of Aside

Ex ctd To find the matrix A_L for L (wrt the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$) we find $L(e_1)$, $L(e_2)$ & $L(e_3)$ and then

$$A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$L(e_1) = n \times e_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= e_1(n_2(0) - n_3(0)) - e_2(n_1(0) - n_3(1)) + e_3(n_1(0) - n_2(1))$$
$$= e_1 \cdot 0 + n_3 e_2 - n_2 e_3$$
$$= (0, n_3, -n_2)$$

So 1^{st} col of A_L is $\begin{pmatrix} 0 & \vdots & \vdots \\ n_3 & \vdots & \vdots \\ -n_2 & \vdots & \vdots \end{pmatrix}$

Next, $L(e_2) = n \times e_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 1 & 0 \end{vmatrix}$

$$= e_1(n_2(0) - n_3(1)) - e_2\underbrace{(n_1(0) - n_3(0))}_{0} + e_3(1n_1 - n_2(0))$$

$$= -n_3 e_1 + 0 e_2 + n_1 e_3$$
$$= (-n_3, 0, n_1)$$

So A_L is $\begin{pmatrix} 0 & -n_3 & \vdots \\ n_3 & 0 & \vdots \\ -n_2 & n_1 & \vdots \end{pmatrix}$ $L(e_3)$