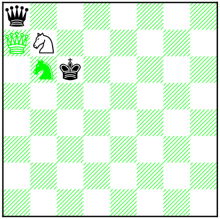

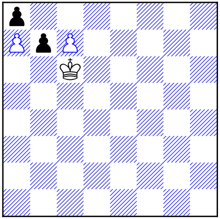

MA284 : Discrete Mathematics

Week 3: Binomial Coefficients

Dr Kevin Jennings

21 & 23 September 2022

Tutorials started this week! (week beginning **19** September).

You should attend *one tutorial per week*.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			MRA 201		(Lecture)
12 – 1		CA117			
1 – 2			(Lecture)		
2 – 3			AC215		
3 – 4		AC213		ADB1020	
4 – 5				AMB-G008	

Last chance to email Kevin now if *none* of these times work for you, with your course details.

ASSIGNMENT 1 is now open!

To access the assignment, go to the 2223-MA284 Blackboard page, select **`Assignments`** ... **`Assignment 1`**.

There are 10 questions.
You may attempt each one up to 10 times.
This assignment contributes approximately 8% to your final grade for Discrete Mathematics.

Deadline: 5pm, Monday 3 October 2022.

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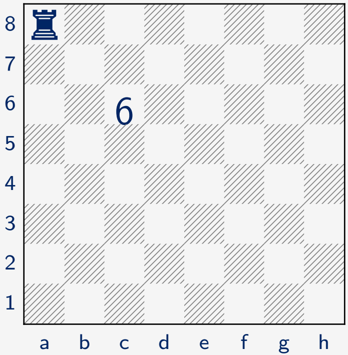
Week 3: Binomial Coefficients

Start of ...

PART 1: Bit strings and lattice paths

A rook can move only in straight lines (not diagonally). *Fill in each square of the chess board below with the number of different shortest paths the rook in the upper left corner can take to get to the square, moving one space at a time.* For example, there are **six** paths from the rook to the square **c6**: DDRR, DRDR, DRRD, RDDR, RDRD, and RRDD. (R = *right*, D = *down*).



A **bit** is a "binary digits" (i.e., 0 or 1).
A **bit string** is a string (list) of bits, e.g. 1001, 0, 111111, 10101010.

The *length* of the string is the number of bits.
A n-bit string has length n.
The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.

Examples:

The *weight* of the string is the number of 1's.
The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

Examples:

Bit strings

- The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.
- The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

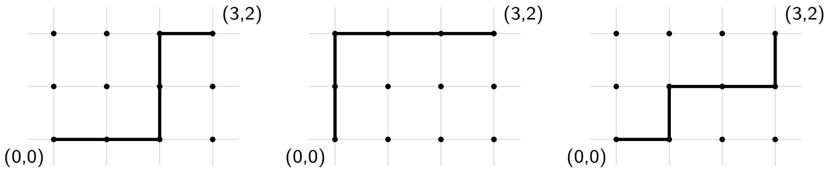
Some counting questions:

1. How many bit strings are there of length 5? That is, what is $|\mathbf{B}^5|$?
2. Of these, how many have weight 3? That is, what is $|\mathbf{B}^5_3|$?

The (integer) *lattice* is the set of all points in the Cartesian plane for which both the x and y coordinates are integers.

A *lattice path* is a **shortest possible path** connecting two points on the lattice, moving only horizontally and vertically.

Example: three possible lattice paths from the points $(0, 0)$ to $(3, 2)$ are:



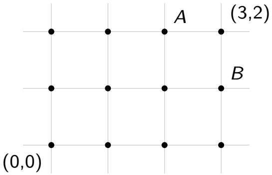
Question: How many lattice paths are there from $(0, 0)$ to $(3, 2)$?

Useful observation 1

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as $|B_3^5|$.
Why?

Useful observation 2

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as the number from $(0,0)$ to $(2,2)$, plus the number from $(0,0)$ to $(3,1)$.



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END OF PART 1

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Start of ...

PART 2: Binomial coefficients

Version 1

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^0 = 1$$
$$(x+y)^1 = x + y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

So, by doing a lot of multiplication, we have worked out that the coefficient of x^3y^2 is 10 (which is rather familiar....)

But, not surprisingly there this a more systematic way of answering this problem.

Version 2

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

We can work out the coefficient of x^3y^2 in the expansion of $(x+y)^5$ by counting the number of ways we can choose three x's and two y's in

$$(x+y)(x+y)(x+y)(x+y)(x+y).$$

These numbers that occurred in all our examples are called *binomial coefficients*, and are denoted $\binom{n}{k}$

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as ``} n \text{ choose } k\text{''}$$

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n, each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

If we were to skip ahead we would learn that there is a formula for

$$\binom{n}{k} \qquad \text{(that is, ``} n \text{ choose } k\text{'')}$$

that is expressed in terms of **factorials**.

Recall that the *factorial* of a natural number, n is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1.$$

Examples:

We will eventually learn that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples

However, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is not very useful in practice.

Example

Suppose there were 200 students in this Discrete Mathematics class, and we want to arrange a tutorial group of 25 students. How many ways could we do this?

Answer: 4.5217×10^{31}. But this is not easy to compute...

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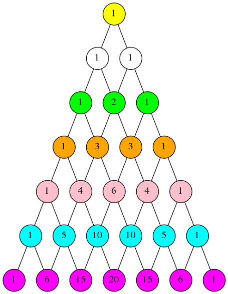
END OF PART 2

MA284
Week 3: Binomial Coefficients

Start of ...

PART 3: Pascal's triangle



Earlier, we learned that if the set of all n-bit strings with weight k is written $\mathbf{B}_k^n$, then

$$|\mathbf{B}_k^n| = |\mathbf{B}_{k-1}^{n-1}| + |\mathbf{B}_k^{n-1}|.$$

Similarly, we get find that...

Pascal's Identity: a recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Why:

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is often presented as *Pascal's Triangle*

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

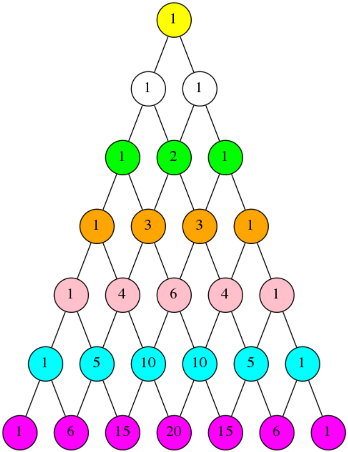
$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

$$\binom{4}{0} \qquad \binom{4}{1} \qquad \binom{4}{2} \qquad \binom{4}{3} \qquad \binom{4}{4}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



Source: http://www.flickr.com/photos/35652310@N00/4139577452/.

Example

The University of Galway Animal Shelter has 4 cats.

(a) How many choices do we have for a single cat to adopt?

(b) How many choices do we have if we want to adopt two cats?

(c) How many choices do we have if we want to adopt three cats?

(d) How many choices do we have if we want to adopt four cats?

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END OF PART 3

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Start of ...

PART 4: Permutations

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Example: List all permutations of the letters `A`, `R` and `T`?

Important: order matters - "ART" $\neq$ "TAR" $\neq$ "RAT".

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

We can also count the number of permutations of the letters `A`, `R` and `T`, without listing them:

More generally, recall that $n!$ (read "n factorial") is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

E.g.,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720.$$

$$10! = 3,628,800, \quad 20! = 2,432,902,008,176,640,000 \approx 2.43 \times 10^{18}.$$

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

To emphasis the **order matters** in permutations, consider the following example.

Example

In last year's paralympics, **8** athletes contested the Men's Va'a 200m Singles's final. How many different finishing orderings were possible?

(Sam Barnes/Sportsfile)

Permutations of k objects from n

The number of permutations of k objects out of n, $P(n, k)$, is

$$P(n, k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Example ($P(8,3)$)

In last year's (delayed) paralympics, **8** athletes contested the Men's Va'a 200m Singles's final. In how many different ways could the gold, silver, and bronze medals be awarded?


TOKYO 2020

SCHEDULE & RESULTS MEDALS SPORTS NEWS VIDEOS ATHLETES TEAMS/NPC FANZONE

Start List Race Results Official Reports

Sea Forest Waterway - 4 Sep - 11:56 - Official

Race Results

Rank	Name	Time	Time Behind
1	AUS McGRATH Curtis Sport Class: VL3	50.537	
2	BRA VIEIRA de PAULA Giovane Sport Class: VL3	52.148	+1.611
3	GBR WOOD Stuart Sport Class: VL3	52.760	+2.223
4	UZB SHERKUZIEV Khaytmurot Sport Class: VL3	52.793	+2.256
5	IRL O'LEARY Patrick Sport Class: VL3	52.910	+2.373
6	FRA POTDEVIN Eddie Sport Class: VL3	53.055	+2.518
7	BRA RIBEIRO de CARVALHO Caio Sport Class: VL3	53.246	+2.709
8	NZL MARTLEW Scott Sport Class: VL3	54.756	+4.219

Choosing the "back three" on a rugby team...

Ireland Squad for the Women's Rugby World Cup Qualifiers had 5 players who (we'll say) could all play on the Left Wing (11), Right Wing (14) or Full-Back (15):

Amee-Leigh Murphy-Crowe • Eimear Considine • Lauren Delany
Beibhinn Parsons • Lucy Mulhall

1. How many choices do we have for picking the starting back three, without assigning them numbers?

   

   Beibhinn Parsons scoring against Italy last year

2. How many choices for picking a starting 11, 14 and 15 (i.e., numbers are assigned)?

Still choosing the back three...

Our rugby squad has 5 backs that can play at 11, 14, or 15.
There are $\binom{5}{3}$ ways we can pick 3 of them for our starting team, without allocating numbers.

Once we have picked these three, there are $3! = 6$ ways we can assign them the 11, 14 and 15 jerseys. That is

$$P(5,3) = \binom{5}{3} 3!.$$

However, we know $P(5,3)$, so this gives a formula for $\binom{5}{3}$.

(1) We know there are $P(n, k)$ permutations of k objects out of n.

(2) We know that

$$P(n, k) = \frac{n!}{(n-k)!}$$

(3) Another way of making a permutation of k objects out of n is to

(a) Choose k from n without order. There are $\binom{n}{k}$ ways of doing this.

(b) Then count all the ways of ordering these k objects. There are $k!$ ways of doing this.

(c) By the Multiplicative Principle,

$$P(n, k) = \binom{n}{k} k!$$

(4) So now we know that $\dfrac{n!}{(n-k)!} = \dbinom{n}{k} k!$

(5) This gives the formula $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$

Q1. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there total?
(b) How many subsets have $\{2, 3, 5\}$ as a subset?
(c) How many subsets contain at least one odd number?
(d) How many subsets contain exactly one even number?
(e) How many subsets are there of cardinality 4?
(f) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?
(g) How many subsets of cardinality 4 contain at least one odd number?
(h) How many subsets of cardinality 4 contain exactly one even number?

Q2. How many subsets of $\{0, 1, \dots, 9\}$ have cardinality 6 or more? (Hint: Break the question into five cases).

Q3. How many shortest lattice paths start at (3,3) and end at (10,10)?
How many shortest lattice paths start at (3,3), end at (10,10), and pass through (5,7)?

Q4. Suppose you are ordering a large pizza from *D.P. Dough*. You want 3 distinct toppings, chosen from their list of 11 vegetarian toppings.

(a) How many choices do you have for your pizza?
(b) How many choices do you have for your pizza if you refuse to have pineapple as one of your toppings?
(c) How many choices do you have for your pizza if you *insist* on having pineapple as one of your toppings?
(d) How do the three questions above relate to each other?