CS4423: Networks

Lecture 6: Connectivity and Permutations

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These slides are by Niall Madden. Elements are based on "A First Course in Network Theory" by Estrada and Knight

Outline

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Slides are at:

https://www.niallmadden.ie/2425-CS4423

Data collection

(Stealing an idea from Angela Carnevale) I'd like to gather some data for use in the class. So, I'm going to run a little survey on what programmes/shows people watch. To do that, I need some ideas... So far we have

1. Only Murders in the Building
2. Breaking Bad
3. The Penguin
4. Succession
5. Squid Game
6. The Bear

Any more?

Notation

- If we write $A = (a_{ij})$ we mean A is a matrix, and a_{ij} is its entry row i, column j.
- We also write such entries as $(A)_{ij}$. The reason for this slightly different notation is that, for example $(A^2)_{ij}$ is the entry in i, column j of $B = A^2$.
- (Very standard) The **trace** of a matrix is the sum of its diagonal entries. That is $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.
- When we write $A > 0$, we mean all entries of A are positive.

Counting Walks

Recall... the **adjacency matrix** of a graph, G of order n, is a square $n \times n$ matrix, $A = (a_{ij})$, with rows and columns corresponding to the nodes of the graph. Set a_{ij} to be the number of edges between nodes i and j.

We learned yesterday that,

- If e_j is the Jth column of the I_n, then $(Ae_j)_i$ is the number of walks of length 1 from node i to node j. (Also, it is just a_{ij}...)
- Moreover, $\big(A(Ae_j)\big)_i = \big(A^2 e_j\big)_i$ is the number of walks of length 2 from node i to node j. We can conclude that, if $B = A^2$, then $b_{i,j}$ is the number of walks of length 2 between nodes i and j.
  Note: b_{ii} is the degree of node i.
- **IN FACT** if $B = A^k$, then $b_{i,j}$ is the number of walks of length k between nodes i and j.

Counting Walks

Example: K_{22}

Paths

Definition (Trail)

A **trail** is a walk with no repeated edges.

Definition (Cycle and triangle)

A **cycle** is a trail in which the first and last nodes are the same, but no other node is repeated. A **triangle** is a cycle of length 3.

Definition (Path)

A **path** is walk with no nodes (and so no edges) repeated. (The idea of a **path** is hugely important in network theory. We'll return to it often)

Path length and shortest path

The **length** of a path is the number of edges in it.
A path from node u to node v is a **shortest** path, if there is no path between them that is shorter (though there could be other paths of the same length)

Finding shortest paths in a network is a major topic in networks, and one we'll return to at another time. But, for now, we'll see how to use powers of the adjacency matrix to find the length of such a part (without finding the path itself).

Some facts about **walks** and **paths**

- Every path is also a walk.
- If a particular walk is the shortest walk between two nodes, then it is also the shortest path between those two nodes.
- If k is the smallest natural number for which $(A^k)_{ij} \neq 0$, then the shortest walk from node i to node j is of length k.
- It follows that k is also the length of the shortest path from Node i to node j.

Example: Consider the graph (see board) with adjacency matrix, and its powers:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 4 & 1 & 1 \\ 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 4 & 2 & 3 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$$

In the previous example, we observed that, where A is the adjacency matrix of the graph G,

- $(A^2)_{ii}$ is the degree of node i.
- $\mathrm{tr}(A^2)$ is the degree sum of the nodes in G.
- $(A^3)_{ii} \neq 0$ if node i is in a triangle.
- $\mathrm{tr}(A^3)/6$ is the number of triangles in G.
- If G is bipartite, then $(A^3)_{ij} = 0$ for all i, j

Connectivity

Let G be a graph, and A its adjacency matrix.

Definition (Reach)

In G, Node i can be **reached** from Node j is there is a path between them.

Fact

If Node i is reachable from Node j, then $(A^k)_{i,j} \neq 0$ for some k.
Also, note that $k \leq n$.
Equivalently, since each power of A is nonnegative, we can say that $(I + A + A^2 + A^3 + \cdots + A^k) > 0$.

Connectivity

Definition (Connected Graph)

A graph/network is **connected** if there is an path between every pair of nodes. That is, every node is reachable from every other. If the graph is *not* connected, we say it is **disconnected**.

Determining if a graph is connected is important. (We'll see later, this is especially important/interesting with *directed graphs*).

Fact

A graph, G of order n is connected if, and only if, for each i, j, there is some $k \leq n$ for which $(A^k)_{i,j} \neq 0$.

Connectivity

Example

Sketch the graph, G, on the nodes $\{1, 2, 3, 4, 5\}$ with edges $1-3$, $1-4$, $2-5$, $3-4$. Write down its adjacency matrix. Is G connected?

Permutation matrices

We know that the structure of a network is not changes by relabelling its nodes. Sometimes, it is is useful to relabel them in order to expose certain properties, such as connectivity.

Example:

Since we think of the nodes as all being numbered from 1 to n, this is the same as **permuting** the numbers of some subset of the nodes.

Permutation matrices

When working with the adjacency matrix of a graph, such a permutation is expressed in terms of a **permutation matrix**, P: this is a $0-1$ matrix (a.k.a. a "Boolean" or "binary" matrix), where there is a single 1 om every row and column.

If the nodes of a graph G (with adjacency matrix A) are listed as entries in a vector, q, then

- Pq is a permutation of the nodes, and
- PAP^T is the adjacency matrix of the graph with that node permutation applied.

Permutation matrices are important when studying graph connectivity because...

FACT!

A graph with adjacency matrix A is **disconnected** if and only if there is a permutation matrix P such that

$$A = P \begin{pmatrix} X & O \\ O^T & Y \end{pmatrix},$$

where O represents the zero matrix with the same number of rows as X and the same number of columns as Y.

Example:

Exercise(s)

1. Write down A, the adjacency matrix of C_5. Try to write down A^2 and A^3 simply by looking at the network it represents.
2. Let u be a vector with n entries. Let $D = \mathrm{diag}(u)$. That is, $D = (d_{ij})$ is the diagonal matrix with entries

   $$d_{ij} = \begin{cases} u_i & i = j \\ 0 & i \neq j. \end{cases}$$

   Verify that $PDP^T = \mathrm{diag}(Pu)$.
3. In all the examples we looked at, we had a symmetric P. Is every permutation matrix symmetric? If so, explain why. If not, give an example.