$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longrightarrow L(v)$$

Recap: L linear if

- $L(u+v) = L(u) + L(v) \quad \forall u,v \in \mathbb{R}^n$
- $L(rv) = rL(v) \quad \forall v \in \mathbb{R}^n, \forall r \in \mathbb{R}$

Then $L \longleftrightarrow$ matrix A where $A = \begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \cdots & L(e_n) \\ | & | & & | \end{pmatrix}$

and $L(v) \Longleftrightarrow Av$

is a $m \times n$ matrix (with respect to standard bases in $\mathbb{R}^n$ & $\mathbb{R}^m$)

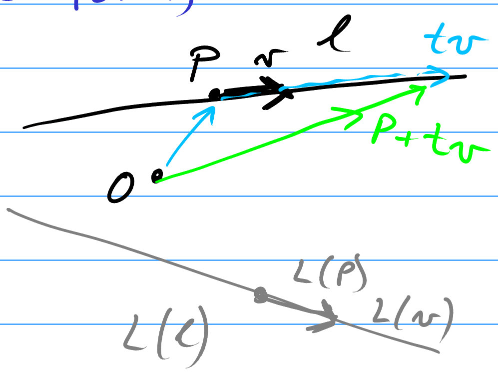
Observe:

i) L maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$

Because
$$\underbrace{L(0+0)}_{\parallel} = L(0)$$
$$L(0) + L(0) = 2(0)$$
$$\Rightarrow \quad L(0) = 0$$

Alternatively using the matrix $A \Longleftrightarrow L$
$L(0)$ is obtained as $A \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\in \mathbb{R}^n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m$

* ii) A linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ sends lines to lines because given a line ℓ in $\mathbb{R}^n$ (in parametric form)
$$\ell : P + tv, \quad t \in \mathbb{R}$$
then
$$L(\ell) = L(P + tv)$$
$$= L(P) + L(tv)$$
$$= L(P) + tL(v)$$



This is the parametric form of the line (in $\mathbb{R}^m$) through the point $L(P)$ in the direction $L(v)$

?? (and $L(v) \neq 0$ if $v \neq 0$)

Examples of Linear Transformations

Ex: Fix a vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$

Define $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as follows:

$$L : v \longrightarrow n \times v$$

($\times$ = vector cross product)

Exercise: Check that L is linear ie

$$n \times (v + w) = n \times v + n \times w$$

$$\& \quad n \times (kv) = k(n \times v) \qquad k \in \mathbb{R}$$

Aside: Let $u = (u_1, u_2, u_3)$
$v = (v_1, v_2, v_3)$

Then $u \times v$

"u cross v" $= (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$

Then $u \times v$ is $\perp^r$ to u
& is $\perp^r$ to v (assuming $u \not\parallel v$)

Check: $(u \times v) \cdot v$ [should be 0 if they are $\perp^r$]

$$(u \times v) \cdot v = u_2 v_3 v_1 - u_3 v_2 v_1 + u_3 v_1 v_2 - u_1 v_3 v_2 + u_1 v_2 v_3 - u_2 v_1 v_3$$

$$= 0$$

[Exercise: Check that $(u \times v) \cdot u = 0$ too]

Mnemonic for $u \times v$

$u = u_1 e_1 + u_2 e_2 + u_3 e_3$

$e_1 = (1, 0, 0)$ etc

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

determinant is $e_1(u_2v_3 - u_3v_2) - e_2(u_1v_3 - u_3v_1)$
(see later) $+ e_3(u_1v_2 - u_2v_1)$

$= (u_2v_3 - u_3v_2,\ u_1v_3 - u_3v_1,\ u_1v_2 - u_2v_1)$

End of Aside

Ex ctd To find the matrix A_L for L (wrt the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$) we find $L(e_1)$, $L(e_2)$ & $L(e_3)$ and then

$$A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$L(e_1) = n \times e_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= e_1(n_2(0) - n_3(0)) - e_2(n_1(0) - n_3(1)) + e_3(n_1(0) - n_2(1))$$
$$= e_1 \cdot 0 + n_3 e_2 - n_2 e_3$$
$$= (0, n_3, -n_2)$$

So 1^{st} col of A_L is $\begin{pmatrix} 0 & \vdots & \vdots \\ n_3 & \vdots & \vdots \\ -n_2 & \vdots & \vdots \end{pmatrix}$

Next, $L(e_2) = n \times e_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 1 & 0 \end{vmatrix}$

$$= e_1(n_2(0) - n_3(1)) - e_2\underbrace{(n_1(0) - n_3(0))}_{0} + e_3(1n_1 - n_2(0))$$

$$= -n_3 e_1 + 0 e_2 + n_1 e_3$$
$$= (-n_3, 0, n_1)$$

So A_L is $\begin{pmatrix} 0 & -n_3 & \vdots \\ n_3 & 0 & \vdots \\ -n_2 & n_1 & \vdots \end{pmatrix}$ $L(e_3)$

Lect 12:

Finally $L(e_3) = n \times e_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 0 & 1 \end{vmatrix}$

$$= e_1(n_2(1) - n_3(0)) - e_2(n_1(1) - n_3(0)) + e_3(n_1(0) - n_2(0))$$
$$= n_2 e_1 - n_1 e_2 + 0 e_3$$
$$= (n_2, -n_1, 0)$$

So 3^{rd} col. of A_L is $\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$

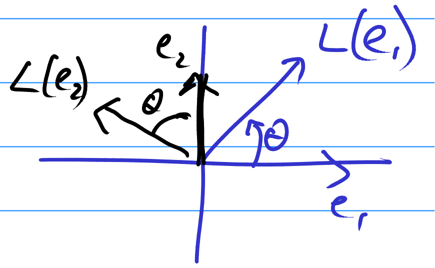
(This is a skew symmetric matrix ie $A^T = -A$)

Recall from last year the matrix A_L for a rotation L in $\mathbb{R}^2$ about the origin $\circlearrowleft$ by an angle θ

We need to find $L(e_1) = L((1,0))$
$L(e_2) = L((0,1))$

and then $A_L = \begin{pmatrix} \uparrow & \uparrow \\ L(e_1) & L(e_2) \\ \downarrow & \downarrow \end{pmatrix}$

We see that (exercise)
$L(e_1) = (\cos\theta, \sin\theta)$
& $L(e_2) = (-\sin\theta, \cos\theta)$



So $A_L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Ex:

Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the e_3 axis $\circlearrowleft$ by an angle θ.

Then in $e_1 - e_2$ plane we see that

$$L(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2 + 0 e_3$$

& so the first column of A_L is:

$$\begin{pmatrix} \cos\theta & \cdot & \cdot \\ \sin\theta & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix}$$

and $L(e_2) = -\sin\theta\, e_1 + \cos\theta\, e_2 + 0 e_3$

so the 2^{nd} col. of A_L is

$$\begin{pmatrix} \cos\theta & -\sin\theta & \cdot \\ \sin\theta & \cos\theta & \cdot \\ 0 & 0 & \cdot \end{pmatrix}$$

and finally $L(e_3) = e_3$

$$= 0e_1 + 0e_2 + 1e_3$$

& our 3^{rd} column is

$$\begin{pmatrix} \vdots & \vdots & 0 \\ \vdots & \vdots & 0 \\ \vdots & \vdots & 1 \end{pmatrix}$$

ie $A_L = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Similarly the matrix for a rotation about the e_1 axis is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

(check this!)

Ex Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be projection of $\mathbb{R}^3$ onto the x_1-x_2 plane.

ie $L((x_1, x_2, x_3)) = (x_1, x_2)$

So $L((1,0,0)) = (1,0)$
$L((0,1,0)) = (0,1)$
& $L((0,0,1)) = (0,0)$

So $A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

§ Revision of Matrix Multiplication

Recall that matrix multiplication was so defined so that if

$L_1 \Longleftrightarrow A$ & $L_2 \Longleftrightarrow B$

Then $L_1 \circ L_2 \Longleftrightarrow AB$ ($\neq BA$ usually)

So if $v = (x_1, x_2, \dots, x_n)$ to find $L_1 \circ L_2(v) = L_1(L_2(v))$ we place $v = (x_1, \dots, x_n)$ behind the matrix AB as a column and multiply

$\left\{ \begin{array}{l} L_2 : \mathbb{R}^n \to \mathbb{R}^m \\ \& \\ L_1 : \mathbb{R}^m \to \mathbb{R}^p \end{array} \right.$

$$\underset{p \times m}{A} \underset{m \times n}{B} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1} := \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}_{p \times 1}$$

& $L_1 \circ L_2(v) = (y_1, y_2, \dots, y_p)$

Eg If $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}$ & $B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

Then AB does not make sense but BA does.