MA284 : Discrete Mathematics

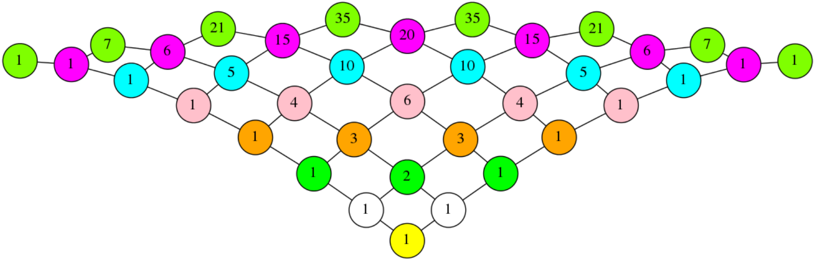
Week 1: Intro to Discrete Mathematics; The Additive and Multiplicative Principles

delivered by Kevin Jennings (Kevin.Jennings@UniversityOfGalway.ie)

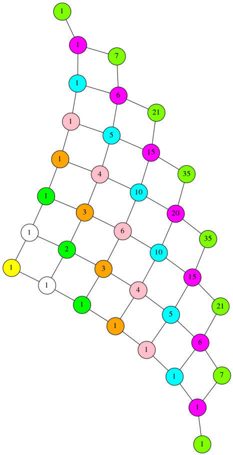
with thanks to Dr Niall Madden who prepared the material and notes

Any mistakes or typos are Kevin's ... any bad jokes have multiple parents ...

7 & 9 September, 2022



Outline



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Week 1: Intro to Discrete Mathematics; The Additive and Multiplicative Principles

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PART 1: All about MA284

Lecturer: **Dr Kevin Jennings** (he/him),
School of Mathematical and Statistical Science, University of Galway.
Room ADB-G008, Árus de Brún
Email: `Kevin.Jennings@UniversityOfGalway.ie`,

The best way to contact me is by email (or possibly using the chat feature on Blackboard).

https://commons.wikimedia.org/wiki/File:%C3%81ras_de_Br%C3%BAn.jpg

This module is taken by about 224 (and counting) students in

- 2nd Science: Mathematics, Mathematical Science, Physics, E&O, Computer Science, Financial Mathematics and Economics, ...;
- Arts: 2nd Mathematics, 2nd Music(!), 3rd Mathematics/Computer Science & Education, Data Science...;
- 2nd Computer Science & IT (2BCT1);
- Visiting student(s).

Given your *very* varied backgrounds, you will need to stay focused, and become practised at communicating your own insights and challenges...

This is *Discrete Mathematics*: a mathematics module introduces the concepts of

- *enumerative combinatorics*: how to count,
- *graph theory* (i.e., the theory of graphs).

Don't worry: most of the rest of the definitions in this module will be more helpful than that!

Lectures: Wednesdays, **13.00–13.50** Anderson Theatre
Fridays, **11.00–11.50** Fottrell Lecture Theatre.

Tutorials: They will start in Week 3. More details in a moment.

Blackboard: You will find lots of resources on Blackboard

- Announcements;
- These slides;
- Grades;
- Link to the textbook,
- Access to assignments.

Work load: 5 ECTS (60 is the typical total for a full-time programme)
24 lectures, all in Semester 1
Roughly 120 hours of student effort time.

Lecture materials: Slides for the week's classes will be available for download in advance of the Wednesday lecture. Please let me know if you spot typos or if the slides are inaccessible in any way (eg to screen-readers).

The slides contain the main definitions, ideas, and examples. Examples that are worked out in class will be posted later in the week.

Each set of slides finishes with a list of exercises, which are of a similar style and standard as those on the final exam.

Images: Particularly in the second half of this module, there will be lots of pictures of graphs. These are mostly generated using **Graphviz** `http://www.graphviz.org/` and/or NetworkX `https://networkx.github.io/`

I'll make Dr Madden's source code available. But if I forget, please ask!

SUMS: The School of Maths provides a free drop-in centre called

SUMS: Support for Undergraduate Maths Students.

SUMS opens from **2pm to 5pm, Monday to Friday**, from Monday of Week 3. For more information, see
http://www.maths.nuigalway.ie/sums/

Devices: The use of portable electronic devices during class is *encouraged*. For example, you might want to use it to check Wikipedia, or access the textbook.
Be aware that these can be distracting to other students. Please be considerate.

Other stuff: Last year this lecture fell on **Soc's Day**! Why not (re)join the Mathematics Society?
https://www.facebook.com/MathsSocNUIG
Also, consider joining our Student Chapter of SIAM:
http://www.maths.nuigalway.ie/SIAM-Galway/

MA284 will be assessed as follows.

Continuous assessment: There will be **five** online assignments, together worth 40% of the final grade.
Multiple attempts can be made, and scoring (right/wrong) is provided immediately. These will help you test yourself, and give you time to seek support at tutorials.

WeBWorK: The Online Assignments uses "**WeBWorK**", the same system as the interactive exercises in the text-book (more about that in a minute). You access the assignments through Blackboard.

Final assessment: There will be a 2 hour exam at the end of the semester, worth **60%**.

Tutorials will start in Week 3 (week beginning **19** September). You should attend *one tutorial per week*.

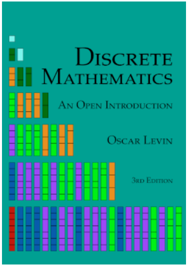
The tentative arrangements for this year below.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			MRA 201		(Lecture)
12 – 1		CA117			
1 – 2			(Lecture)		
2 – 3			AC215		
3 – 4		AC213		ADB1020	
4 – 5				AMB-G008	

Please email Kevin now if *none* of these times work for you, with your course details.

The main recommended text is

Oscar Levin, Discrete Mathematics: an open introduction, 3rd Edition. This is a free, open source textbook, available from http://discretetext.oscarlevin.com, in both printable and tablet/ereader-friendly versions. It is published under Creative Commons (CC BY-SA 4.0)



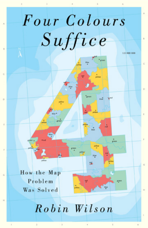
Other recommended texts include:

- Normal L Biggs, *Discrete Mathematics*, Oxford Science Publications. There are about 10 copies in the library at `510 BIG`.
- Kenneth Rosen, *Discrete Mathematics and Its Applications*, McGraw-Hill. Located at `511 ROS`.

Other books and resources will be mentioned through the semester.

Some related, fun, reading.

Really Big Numbers, by Richard Schwartz, published by the American Mathematical Society.
It is aimed a children, but is quite sophisticated. So you can learn some Discrete Mathematics while doing bed-time reading! It is in the library at 513 SCH
Watch at
`https://www.youtube.com/watch?v=cEOY9UAsCFM`



Four Colors Suffice: How the Map Problem Was Solved.
Robin Wilson.
In the library at 511.5 WIL
This is the story of the solution of one of most famous mathematical problems, that defied solution for nearly 150 years. It is also a treatise on what "proof" really means.

Do you have any other suggestions?

There are very few prerequisites for this module. I will expect that

- you can reason logically;
- understand the concept of a *proof*, and know several proof techniques, such as *induction*.
- know what a matrix is, and how to multiply a matrix by a vector, and a matrix by a matrix.
- you are comfortable with the concept of **sets**, and the notation used to describe and manipulate them.
- you are comfortable with the concept of **functions**, and the notation used to describe and manipulate them.

Exercise

Read Sections 0.3 (Sets) and 0.4 (Functions) in Chapter 0 of *Discrete Mathematics: an open introduction*

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PART 2: What is Discrete Mathematics

If calculus is "continuous mathematics", then "discrete mathematics" is everything else! However, it is usually taken to include the following

1. Logic
2. Sets and set-theory;
3. Mathematics of Algorithms;
4. Recursion and induction;
5. Counting;
6. Discrete probability;
7. Graphs, trees and networks;
8. Boolean algebra;
9. Modelling computing (Turing machines and Finite State Machines).

But we will just focus on **counting (combinatorics)** and *graphs*.

1. Combinatorics.

How to count,

The additive and multiplicative principles.

The Binomial coefficients and some identities.

The Principle of Inclusion-Exclusion.

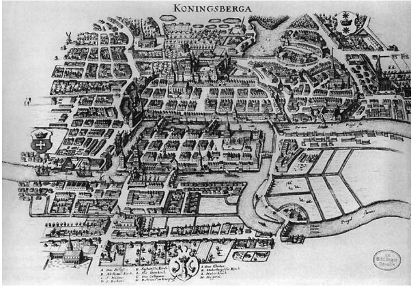
Permutations and Combinations.

Non-negative equations and inequalities.

Derangements and distributions.

2. Graph Theory.

Euler and the Koenigsberg Bridges Problem.
Eulerian and Hamiltonian graphs.
Tree graphs and bipartite graphs.
Planarity of Graphs.
Euler's formula for a connected planar graph.
Planarity and the Platonic solids.
Colouring of Graphs.



Combinatorics has an ancient history. The earliest known is in a 3,500 year old Egyptian manuscript. It posed a question like *"In 7 houses are 7 cats, each with 7 mice, who each have 7 heads of wheat, which each have 7 grains. How many houses, cats, mice, heads of wheat and grains are there?*

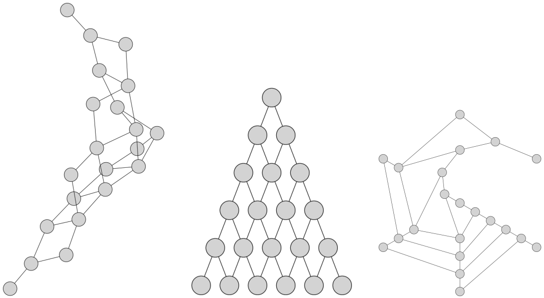
Description: The so-called "Rhind Mathematical Papyrus" : detail (British Museum, EA10057)

Source: `http://www.archaeowiki.org/Image:Rhind_Mathematical_Papyrus.jpg`

Slightly more recently, in the 6th century the Indian physician Sushruta determined that there are $2^6 - 1 = 63$ different combinations of the tastes *sweet*, *pungent*, *astringent*, *sour*, *salt*, and *bitter*.

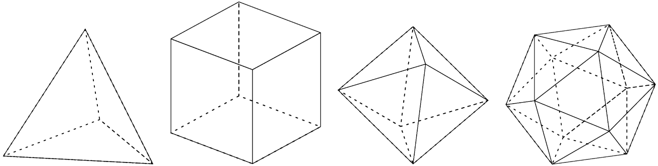
We'll solve problems like the two above, and also:

1. What are your chances of winning the Irish Lottery ("Lotto"). That is, what is the probability of correctly selecting **6** numbers from **47**?
2. If 500,000 people play the Lotto per week. What is the chance of a roll-over (i.e., nobody winning)?
3. For last night's men's European soccer match between Glasgow Celtic and Real Madrid, a **23**-man squad was named for Celtic.
   How many different ways were there of selecting the 11 starting players for the match?
   How many ways could one select (up to 5) of these players to be substituted during the game?
4. My password has 10 characters. Each character is an upper- or lower-case letter, or a digit. How long would it take you to crack my account?

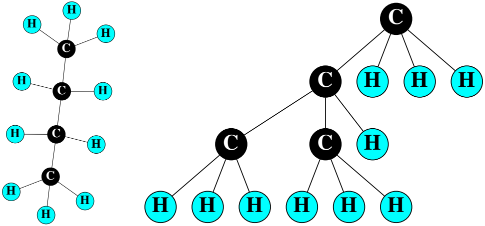


1. Which of these graphs are the **same** (and what does that mean)?
2. Is it possible to draw all the graph on the left so that none of its edges intersect?
3. What is the smallest number of colours needed to colour every vertices so that no two adjacent vertices have the same colour?
4. Is there a "route" through the graph that visits every vertex once and only once?

5. How may regular polyhedra (platonic solids) are there?



6. Are all the graphs of *saturated hydrocarbon isomers* trees?



The most important reason for taking this module is that **Discrete mathematics is one of the most appealing, elegant, and applicable areas of mathematics**.

Appealing: The problems that we will consider are, I believe, easily motivated, but not trivial.

Elegant: The solutions to these problems involve some clever reasoning, but never tedious calculations.

Applicable: In spite of its classical origins, graph theory is one of the hottest topics in both pure and applied mathematics, with applications to network science, computer science, linguistics, chemistry, physics, biology, social science, and music.

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PART 3: Counting

Combinatorics is the mathematics of *counting*. It is an ancient field of study, though its "modern" history began with the systematic study of gambling in the 17^{th} century.

The simplest method of counting is *simple enumeration* = *"Point and count"*.

1. How many students in this class have a last name that begins with `A`?

2. How many anagrams are there of the letters **`NUI`**?

   Usually we don't want to make a list of all possibilities:

3. How many car licence plates are there of the form `XXX-yyy`, where X is a letter and y is a digit?
   Answer: There are 17,576,000, but we don't want to list them all.

The first techniques that we will study for solving counting problems are called

The Additive and Multiplicative Principles

For more information see Chapter 1 (Counting) of Oscar Levin's *Discrete Mathematics: an open introduction*.

1. There are 5 starters and 6 main-courses on a restaurant's menu. How many choices do you have if
   (a) You would like one dish: a starter *or* a main-course?
   (b) You would like two dishes: a starter **and** a main-course?

2. A standard deck of cards has 26 red cards, and 12 face/court cards.
   (a) How many ways can you select a card that is red *and* face card?
   (b) How many ways can you select a card that is red *or* face card?

Think about these questions as we go through the next sections.

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PART 4: The Additive Principle

Example

The University of Galway Animal Shelter has 4 cats and 6 dogs in need of a home. You would like a new pet (but just one!). How many choices do you have?

The Additive Principle

If event A can occur m ways, and event B can occur n (disjoint) ways, then event "A **or** B" can occur in $m + n$ ways.

Example

1. Can we use the additive principle to determine how many two letter "words" **begin with** either A **or** B?

2. Can we use the additive principle to determine how many two letter "words" **contain** either A **or** B?

The Additive Principle

If event A can occur m ways, and event B can occur n (disjoint) ways, then event "A **or** B" can occur in $m + n$ ways.

Example

The University of Galway Animal Shelter has 4 cats, 6 dogs, and 7 donkeys in need of a home. How many choices do you have for a new pet?

The Additive Principle

If event A can occur m ways, and event B can occur n **disjoint** ways, then event "A **or** B" can occur in $m + n$ ways.

Example

A deck of cards has 26 red cards and 12 "face"-cards.

1. How many ways can you pick a red card?
2. How many ways can you pick a face-card?
3. How many ways can you pick a card that is red *or* is a face-card?

This last example is important because it emphasises the importance of the sets being **disjoint**.

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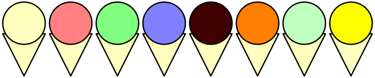
PART 5: The Multiplicative Principle



Example

Your favourite ice-cream shop has **8** flavours of ice-cream.

You can also choose between a cone, a waffle, and a cup.

How many choices to you have?

The Multiplicative Principle

If event A can occur m ways, and each possibility allows for B to occur in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

Example

The University of Galway Animal Shelter has 4 cats and 6 dogs in need of a home. How many choices do you have if you would like a cat and a dog as pets?

Example

The University of Galway Animal Shelter also has 7 donkeys. How many choices to you have if you want a cat, a dog and a donkey?

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PART 6: Counting with sets

A set is a collection of things. The items in a set are called *elements*.
Examples:

- The set of natural numbers from 1 to 10 is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- The set of upper-case letters is $\{A, B, \ldots, Y, Z\}$
- The set of students registered for Discrete Mathematics has 224 elements.
- A set is *unordered*.

You should be familiar with the following basic elements of set notation:

$$\{\cdot\} \quad \in \quad \notin \quad \subseteq \quad \cup \quad \cap \quad \emptyset \quad |\cdot| \quad \backslash$$

Example

Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4\}$.

- $2 \in A$, $\qquad 4 \notin A$
- $\{1, 3\} \subseteq A$
- $A \cup B = \{1, 2, 3, 5\}$
- $A \cap B = \{1, 3\}$, $\qquad B \cap C = \emptyset$,
- $|A| = 3$, $\qquad |B \cap C| = 0$,
- $A \backslash B = \{2\}$ $\qquad A \backslash C = \{1, 3\}$

Also, for any set X, $\quad X \subseteq X \quad \emptyset \subseteq X$.

Let's return to the restaurant problem again, changed slightly...

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PART 7: Exercises

Here are a set of exercises to help you work through the material presented during this week's classes.

All but the last are taken either directly from the textbook, or with minor edits.

You do not have to submit your solutions to be graded.

0. Read Chapter 0 of Levin's *Discrete Mathematics: an open introduction* from `http://discretetext.oscarlevin.com`.
   Do Exercises 1–9 in Chapter 0 (these are interactive, with hints and solutions).
1. Your wardrobe consists of 5 shirts, 3 pairs of pants, 17 bow ties, and one fez (hat). How many different outfits can you make?
2. For your job interview at the University of Galway Animal Shelter, you must wear a tie. You own 3 regular (boring) ties and 5 (cool) bow ties. How many choices do you have for your neck-wear?

3. You realise that the interview is actually for ClownSoc, so you should probably wear both a regular tie and a bow tie. How many choices do you have now?

4. Your DVD collection consists of 9 comedies and 7 horror movies. Give an example of a question for which the answer is:
   (a) 16.
   (b) 63.

5. If $|A| = 10$ and $|B| = 15$, what is the largest possible value for $|A \cap B|$? What is the smallest? What are the possible values for $|A \cup B|$?

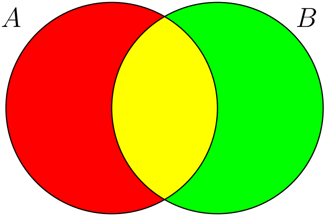
6. If $|A| = 8$ and $|B| = 5$, what is $|A \cup B| + |A \cap B|$?

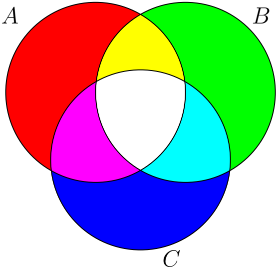
MA284 : Discrete Mathematics

Week 2: Counting with sets and the PIE

Dr Kevin Jennings

14 & 16 September, 2022





Tutorials

Tutorials will start next week (week beginning Monday, 19 September).

You should attend *one tutorial per week*.

The proposed tutorial times are

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			MRA 201		(Lecture)
12 – 1		CA117			
1 – 2			(Lecture)		
2 – 3			AC215		
3 – 4		AC213		ADB1020	
4 – 5				AMB-G008	

Please email Kevin now if *none* of these times work for you, with your course details.

SUMS also opens next week. Dr Kirsten Pfeiffer will be here next week to give more information.

We will use WeBWorK for all assignments in this module. You can access them by logging on to Blackboard, clicking on `Assignments`, and then the relevant link.

At present (14 Sep) , there is just a `Demo Assignment` there. Please try it out, and report any problems. There are 10 questions, and you may attempt each one up to 10 times.
This problem set does **not** contribute to your CA score for MA284.

The first proper assignment will open on Friday.

In this week's classes, we are going to build on the *Additive* and *Multiplicative* Principles from Lecture 2.
After reminding ourselves of the basic ideas, we will present them in the formal setting of *set theory*.

We will then move on to the *Principle of Inclusion/Exclusion* (PIE).
The presentation will closely follow Chapter 1 of Levin's *Discrete Mathematics: an open introduction*.

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PART 1: Review from Week 1

The Additive Principle

If Event A can occur m ways, and Event B can occur n (disjoint) ways, then Event "A **or** B" can occur in $m + n$ ways.

Example

There are (now) **235** students in registered for Discrete Mathematics, of which **60** are in Financial Maths & Economics (FM), **55** are in Arts, and the remaining **120** are in various Sciences (including Computer Science).

1. In how many ways can we choose a Class Rep who is from **Arts** or **FM**?
2. How many ways can be chosen a Class Rep who is from **Arts**, **FM**, or **Science**?

The Multiplicative Principle

If Event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then Event "A **and** B" can occur in $m \times n$ ways.

Example

There are (still) **235** students in registered for Discrete Mathematics, of which **60** are in Financial Maths & Economics (FM), **55** are in Arts, and the remaining **120** are in various Sciences (including Computer Science).

1. In how many ways can we choose two Class Reps, one each from **Arts** and **FM**?
2. How many ways can we choose three Class Reps, one each from **Arts**, **FM**, and **Science**?

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PART 2: Counting with Sets

Example (Students in Discrete Mathematics (again))

Let D be the set of students in Discrete Mathematics. So $|D| = 235$.
Let F be the set of Discrete Maths students who are in **Financial Maths**. So $|F| = 60$.
Similarly, let S and A be the sets of Discrete Mathematics students who are in **Science** and **Arts** respectively. So $|S| = 120$, and $|A| = 55$.

What do we mean by...

- $A \cup F$?
- $A \cap F$?

Additive Principle in terms of "events"

If Event A can occur m ways, and Event B can occur n (disjoint/ independent) ways, then event "A **or** B" can occur in $m + n$ ways.

But an "event" can be expressed as just selecting an element of a set. For example, the event "*Choose a Class Rep from Arts*" is the same as "*Choose an element of the set A*". Similarly:

- **Event A can occur m ways**, is the same as saying $|A| = m$;
- **Event B can occur n ways**, is the same as saying $|B| = n$;
- Events A and B are disjoint/independent means $|A \cap B| = 0$ (or, equivalently $A \cap B = \emptyset$).

Additive Principle for Sets

Given two sets A and B with $|A| = m$, $|B| = n$ and $|A \cap B| = 0$. Then

$$|A \cup B| = m + n.$$

Additive Principle for Sets

Given two sets A and B with $|A \cap B| = 0$. Then

$$|A \cup B| = |A| + |B|.$$

Example:

The **Cartesian Product** of sets A and B is

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

This is the set of pairs where the first term in each pair comes from A, **and** the second comes from B.

Example

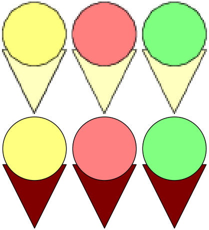
Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4\}$.
Write down $A \times B$ and $A \times C$.

If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.
Why?

What has the *Cartesian Product* got to do with the **Multiplicative Principle**? Consider the following example... Suppose we go to our favourite ice-cream shop where they stock

- three flavours: **V**anilla, **Strawberry** and **M**int.
- two types of cone: plain **C**ones and **W**affle cones.

How many ways can I place an order (for 1 cone and 1 scoop?).



Previously we learned about

The Multiplicative Principle (for events)

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

We can now express this in terms of sets:

Multiplicative Principle for Sets

Given two sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

This extends to three or more sets in the obvious way:

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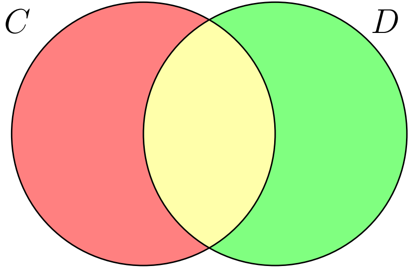
PART 3: The Principle of Inclusion and Exclusion (PIE)

Good news!

Remember from last week that the NUIG Animal Shelter had 4 cats and 6 dogs in need of a home. Well, they have all been adopted and, (unsurprisingly, given their kind and generous nature) by Discrete Mathematics students. They went to 9 different homes, because one person adopted both a cat and a dog.



Since we admire those people that adopted an animal so much, we want one of them as our Class Rep. That is we will choose our Class Rep from one of the sets C and D where $|C| = 4$ and $|D| = 6$.

If we were to apply the **Additive Principle** *naïvly*, we would think that we have $|C| + |D| = 10$ choices for our Rep. But of course, we only have $|C \cup D| = 9$ choices.

So, to correctly calculate the cardinality of a pair of sets (with non-zero intersection) we need *the Principle of Inclusion and Exclusion*.

The Principle of Inclusion and Exclusion (for the union of 2 sets)

For any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This extends to larger numbers of sets. For example,

The Principle of Inclusion and Exclusion, for the union of 3 sets

For any finite sets A, B, and C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example (PIE for 2 sets)

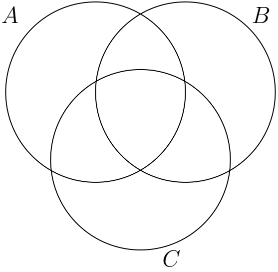
A group of 20 second year maths students are registering for modules. 12 take Discrete Mathematics and, of those, 4 take both Discrete Maths and Differential Forms. If all 20 do at least one of these subjects, how many just take Differential Forms?

Example (See Example 1.1.8 of textbook)

An examination in three subjects, **A**lgebra, **B**iology, and **C**hemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations.

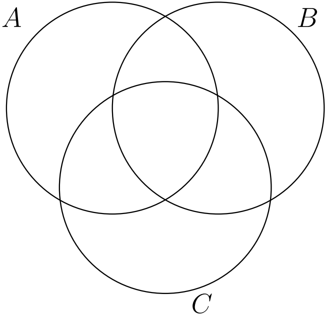
Subject:	A	B	C	A&B	A&C	B&C	A&B&C
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?



This example shows how to extend the PIE to three sets:

$$
\begin{aligned}
|A \cup B \cup C| = |A| + |B| + |C| \\
- |A \cap B| - |A \cap C| - |B \cap C| \\
+ |A \cap B \cap C| \quad .
\end{aligned}
$$



MA284
Week 2: Counting with sets and the PIE

END OF PART 3

MA284
Week 2: Counting with sets and the PIE

Start of ...

PART 4: Subsets & Power Sets

Start here Friday, 16 September

Recall last week it was mentioned that one of the earliest recorded problems in combinatorics is from the *Sushruta Samhita* an ancient Sanskrit text on medicine and surgery.

Palm leaves of the Sushruta Samhita or Sahottara-Tantra from Nepal. Source: https://en.wikipedia.org/wiki/Sushruta_Samhita

The **combinatorics** problem from the Sushruta Samhita is to determine the number of **different possible combinations** of the tastes

(1) *sweet* (2) *pungent* (3) *astringent* (4) *sour*
(5) *salt* and (6) *bitter*.

This is equivalent to the problem of *counting the number of non-empty subsets* there are of a set with 6 elements.

The question we will investigate is:

How many subsets are there of $A_1 = \{1\}$?

How many subsets are there of $A_2 = \{1, 2\}$?

How many subsets are there of $A_3 = \{1, 2, 3\}$?

How many subsets are there of $A_4 = \{1, 2, 3, 4\}$?

$\vdots$

How many subsets are there of $A_k = \{1, 2, 3, \ldots, k\}$?

Here is another way of expressing this:

Power set

The POWER SET of A, denoted by $P(A)$, is the set of all subsets of A, including the empty set.
What is $|P(A)|$?

We'll answer this question in two different ways, which is a classic approach to problems in combinatorics.

First we'll list all the subsets of A_1, A_2 and A_3, and try to guess the answer. Then we will try to explain it.

Here is another approach. Consider $P(A_2) = P(\{1, 2\})$.
When constructing a subset, we can proceed as follows:

- **Event A:** choose to include the element **1** or not. This can happen in 2 ways.
- **Event B:** choose to include the element **2** or not. This can happen in 2 ways.

Now apply the multiplicative principle.

Example

How many subsets are there of $A_5 = \{1, 2, 3, 4, 5\}$?

Here is a slightly harder problem

How many subsets are there of $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

We will look at **three** different ways of answering this question:

1. By "brute-force": simply listing all the possibilities.
2. By counting all sets that **do not** have three elements.
3. Next week, by using **binomial coefficients**.

Method 2

How many subsets are there of $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

Here is an easy way of answering this question.

- How many subsets of A_5 have no elements?
- How many subsets of A_5 have 5 elements?
- How many subsets of A_5 have 1 element?
- How many subsets of A_5 have 4 elements?
- Now use that the number of subsets of A_5 with **3** elements, is the same as the number with **2** elements.

Here are a set of exercises to help you work through the material presented during Week 2.
Except where indicated, all these exercises are taken from Section 1.1 of textbook (Levin's Discrete Mathematics - an open introduction).

1 We usually write numbers in decimal form (i.e., base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes, though, it is useful to write numbers in *hexadecimal* (base 16), which has 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

a. How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).
b. Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?
c. How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.
d. How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

2 A group of students were asked about their TV watching habits. Of those surveyed,

- 28 students watch *The Good Place*,
- 19 watch *Stranger Things*, and
- 24 watch *Orange is the New Black*.

- Additionally, 16 watch *The Good Place* and *Stranger Things*,
- 14 watch *The Good Place* and *Orange is the New Black*,
- and 10 watch *Stranger Things* and *Orange is the New Black*.
- There are 8 students who watch all three shows.

How many students surveyed watched at least one of the shows?

3 (MA284, Final Exam, 2018/2019) In a survey, 36 students were asked if they liked Discrete Mathematics, Statistics and Differential Forms. 16 said they liked Discrete Maths, 20 liked Statistics, 26 admitted to liking Differential Forms, and 1 did not like any. Additionally, 9 students said they liked both Discrete Maths and Statistics, 13 liked Statistics and Differential Forms, and 11 liked Discrete Maths and Differential Forms. How many students like *all* three subjects?

4 In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students do *not* like potatoes? Explain why your answer is correct.

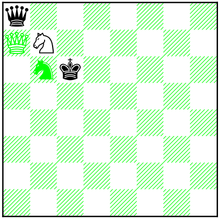
5 (MA284, Semester 1 Exam, 2016/2017) For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

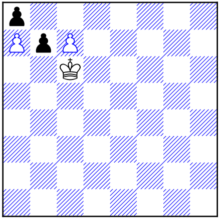
6 Let A, B, and C be sets.

a. Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

b. Describe a set in terms of A, B, and C with cardinality 26.

7 (MA284, Semester 1 Exam, 2017/2018) The sets A and B are such that $|A| = 17$ and $|B| = 9$.
What is the largest possible value of $|A \cup B|$?
What is the smallest possible value of $|A \cup B|$?
What is the largest possible value of $|A \cap B|$?
What is the smallest possible value of $|A \cap B|$?
What is the value of $|A \cup B| + |A \cap B|$?





MA284 : Discrete Mathematics

Week 3: Binomial Coefficients

Dr Kevin Jennings

21 & 23 September 2022

Tutorials started this week! (week beginning **19** September).

You should attend *one tutorial per week*.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			MRA 201		(Lecture)
12 – 1		CA117			
1 – 2			(Lecture)		
2 – 3			AC215		
3 – 4		AC213		ADB1020	
4 – 5				AMB-G008	

Last chance to email Kevin now if *none* of these times work for you, with your course details.

ASSIGNMENT 1 is now open!

To access the assignment, go to the 2223-MA284 Blackboard page, select **`Assignments`** ... **`Assignment 1`**.

There are 10 questions.
You may attempt each one up to 10 times.
This assignment contributes approximately 8% to your final grade for Discrete Mathematics.

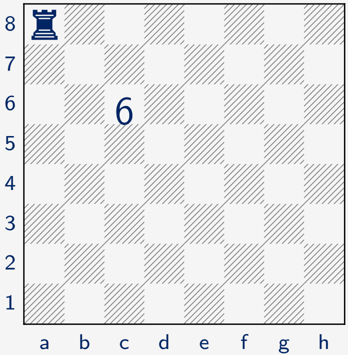
Deadline: 5pm, Monday 3 October 2022.

MA284
Week 3: Binomial Coefficients

Start of ...

PART 1: Bit strings and lattice paths

A rook can move only in straight lines (not diagonally). *Fill in each square of the chess board below with the number of different shortest paths the rook in the upper left corner can take to get to the square, moving one space at a time.* For example, there are **six** paths from the rook to the square **c6**: DDRR, DRDR, DRRD, RDDR, RDRD, and RRDD. (R = *right*, D = *down*).



A **bit** is a "binary digits" (i.e., 0 or 1).
A **bit string** is a string (list) of bits, e.g. 1001, 0, 111111, 10101010.

The *length* of the string is the number of bits.
A n-bit string has length n.
The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.

Examples:

The *weight* of the string is the number of 1's.
The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

Examples:

Bit strings

- The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.
- The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

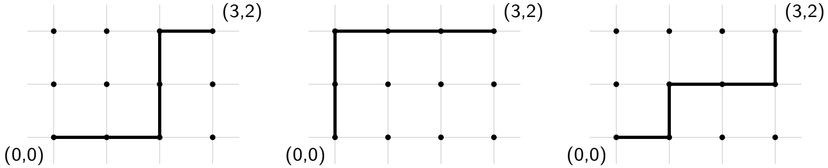
Some counting questions:

1. How many bit strings are there of length 5? That is, what is $|\mathbf{B}^5|$?
2. Of these, how many have weight 3? That is, what is $|\mathbf{B}^5_3|$?

The (integer) *lattice* is the set of all points in the Cartesian plane for which both the x and y coordinates are integers.

A *lattice path* is a **shortest possible path** connecting two points on the lattice, moving only horizontally and vertically.

Example: three possible lattice paths from the points $(0,0)$ to $(3,2)$ are:



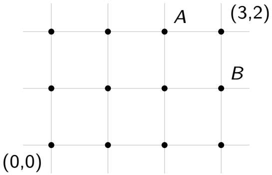
Question: How many lattice paths are there from $(0,0)$ to $(3,2)$?

Useful observation 1

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as $|B_3^5|$.
Why?

Useful observation 2

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as the number from $(0,0)$ to $(2,2)$, plus the number from $(0,0)$ to $(3,1)$.



MA284
Week 3: Binomial Coefficients

END OF PART 1

MA284
Week 3: Binomial Coefficients

Start of ...

PART 2: Binomial coefficients

Version 1

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

So, by doing a lot of multiplication, we have worked out that the coefficient of x^3y^2 is 10 (which is rather familiar....)

But, not surprisingly there this a more systematic way of answering this problem.

Version 2

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

We can work out the coefficient of x^3y^2 in the expansion of $(x+y)^5$ by counting the number of ways we can choose three x's and two y's in

$$(x+y)(x+y)(x+y)(x+y)(x+y).$$

These numbers that occurred in all our examples are called *binomial coefficients*, and are denoted $\binom{n}{k}$

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as "}n\text{ choose }k\text{"}$$

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n, each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

If we were to skip ahead we would learn that there is a formula for

$$\binom{n}{k} \qquad \text{(that is, ``} n \text{ choose } k\text{'')}$$

that is expressed in terms of **factorials**.

Recall that the *factorial* of a natural number, n is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1.$$

Examples:

We will eventually learn that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples

However, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is not very useful in practice.

Example

Suppose there were 200 students in this Discrete Mathematics class, and we want to arrange a tutorial group of 25 students. How many ways could we do this?

Answer: 4.5217×10^{31}. But this is not easy to compute...

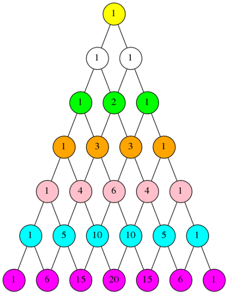
MA284
Week 3: Binomial Coefficients

END OF PART 2

MA284
Week 3: Binomial Coefficients

Start of ...

PART 3: Pascal's triangle



Earlier, we learned that if the set of all n-bit strings with weight k is written $\mathbf{B}^n_k$, then

$$|\mathbf{B}^n_k| = |\mathbf{B}^{n-1}_{k-1}| + |\mathbf{B}^{n-1}_k|.$$

Similarly, we get find that...

Pascal's Identity: a recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Why:

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is often presented as *Pascal's Triangle*

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

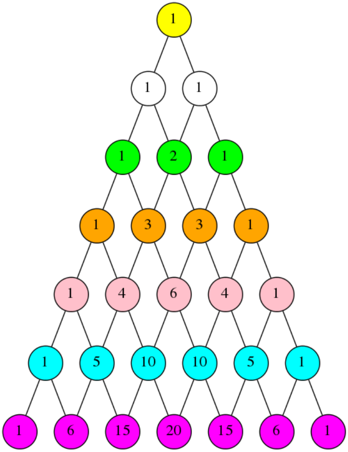
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



Source: http://www.flickr.com/photos/35652310@N00/4139577452/.

Example

The University of Galway Animal Shelter has 4 cats.

(a) How many choices do we have for a single cat to adopt?

(b) How many choices do we have if we want to adopt two cats?

(c) How many choices do we have if we want to adopt three cats?

(d) How many choices do we have if we want to adopt four cats?

MA284
Week 3: Binomial Coefficients

END OF PART 3

MA284
Week 3: Binomial Coefficients

Start of ...

PART 4: Permutations

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Example: List all permutations of the letters `A`, `R` and `T`?

Important: order matters - “ART” $\neq$ “TAR” $\neq$ “RAT”.

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

We can also count the number of permutations of the letters `A`, `R` and `T`, without listing them:

More generally, recall that $n!$ (read "n factorial") is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

E.g.,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720.$$

$$10! = 3,628,800, \quad 20! = 2,432,902,008,176,640,000 \approx 2.43 \times 10^{18}.$$

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

To emphasis the **order matters** in permutations, consider the following example.

Example

In last year's paralympics, **8** athletes contested the Men's Va'a 200m Singles's final. How many different finishing orderings were possible?

(Sam Barnes/Sportsfile)

Permutations of k objects from n

The number of permutations of k objects out of n, $P(n, k)$, is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Example ($P(8, 3)$)

In last year's (delayed) paralympics, **8** athletes contested the Men's Va'a 200m Singles's final. In how many different ways could the gold, silver, and bronze medals be awarded?

TOKYO 2020

SCHEDULE & RESULTS MEDALS SPORTS NEWS VIDEOS ATHLETES TEAMS/NPC FANZONE

Start List Race Results Official Reports

Sea Forest Waterway - 4 Sep - 11:56 - Official

Race Results

Rank	Name	Time	Time Behind
1	AUS McGRATH Curtis Sport Class: VL3	50.537	
2	BRA VIEIRA de PAULA Giovane Sport Class: VL3	52.148	+1.611
3	GBR WOOD Stuart Sport Class: VL3	52.760	+2.223
4	UZB SHERKUZIEV Khaytmurot Sport Class: VL3	52.793	+2.256
5	IRL O'LEARY Patrick Sport Class: VL3	52.910	+2.373
6	FRA POTDEVIN Eddie Sport Class: VL3	53.055	+2.518
7	BRA RIBEIRO de CARVALHO Caio Sport Class: VL3	53.246	+2.709
8	NZL MARTLEW Scott Sport Class: VL3	54.756	+4.219

Choosing the "back three" on a rugby team...

Ireland Squad for the Women's Rugby World Cup Qualifiers had 5 players who (we'll say) could all play on the Left Wing (11), Right Wing (14) or Full-Back (15):

Amee-Leigh Murphy-Crowe • Eimear Considine • Lauren Delany
Beibhinn Parsons • Lucy Mulhall

1. How many choices do we have for picking the starting back three, without assigning them numbers?

Beibhinn Parsons scoring against Italy last year

2. How many choices for picking a starting 11, 14 and 15 (i.e., numbers are assigned)?

Still choosing the back three...

Our rugby squad has 5 backs that can play at 11, 14, or 15.

There are $\binom{5}{3}$ ways we can pick 3 of them for our starting team, without allocating numbers.

Once we have picked these three, there are $3! = 6$ ways we can assign them the 11, 14 and 15 jerseys. That is

$$P(5,3) = \binom{5}{3}3!.$$

However, we know $P(5,3)$, so this gives a formula for $\binom{5}{3}$.

(1) We know there are $P(n, k)$ permutations of k objects out of n.

(2) We know that

$$P(n, k) = \frac{n!}{(n-k)!}$$

(3) Another way of making a permutation of k objects out of n is to

(a) Choose k from n without order. There are $\binom{n}{k}$ ways of doing this.

(b) Then count all the ways of ordering these k objects. There are $k!$ ways of doing this.

(c) By the Multiplicative Principle,

$$P(n, k) = \binom{n}{k} k!$$

(4) So now we know that $\dfrac{n!}{(n-k)!} = \dbinom{n}{k} k!$

(5) This gives the formula $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$

Q1. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there total?
(b) How many subsets have $\{2, 3, 5\}$ as a subset?
(c) How many subsets contain at least one odd number?
(d) How many subsets contain exactly one even number?
(e) How many subsets are there of cardinality 4?
(f) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?
(g) How many subsets of cardinality 4 contain at least one odd number?
(h) How many subsets of cardinality 4 contain exactly one even number?

Q2. How many subsets of $\{0, 1, \ldots, 9\}$ have cardinality 6 or more? (Hint: Break the question into five cases).

Q3. How many shortest lattice paths start at (3,3) and end at (10,10)?
How many shortest lattice paths start at (3,3), end at (10,10), and pass through (5,7)?

Q4. Suppose you are ordering a large pizza from *D.P. Dough*. You want 3 distinct toppings, chosen from their list of 11 vegetarian toppings.

(a) How many choices do you have for your pizza?
(b) How many choices do you have for your pizza if you refuse to have pineapple as one of your toppings?
(c) How many choices do you have for your pizza if you *insist* on having pineapple as one of your toppings?
(d) How do the three questions above relate to each other?

MA284 : Discrete Mathematics

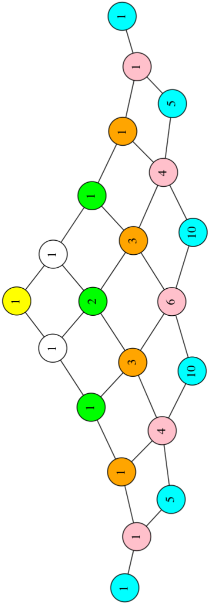
Week 4: Algebraic and Combinatorial Proofs

Dr Kevin Jennings

28 September and & 30 September, 2022

1. Part 1: A short summary
   - Binomial coefficients
2. Part 2: Pascal's Triangle (again)
3. Part 3: Algebraic and Combinatorial Proofs
4. Part 4: How combinatorial proofs work
   - Which is better?
5. Exercises

These slides are based on §1.3 and §1.4 of Oscar Levin's *Discrete Mathematics: an open introduction*.



Tutorials started last week. You should attend one of the sessions listed below.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			MRA 201		(Lecture)
12 – 1		CA117			
1 – 2			(Lecture)		
2 – 3			AC215		
3 – 4		AC213		ADB1020	
4 – 5				AMB-G008	

ASSIGNMENT 1 is open (and ASSIGNMENT 2 opens soon!)

To access the assignment, go to the 2122-MA284 Blackboard page, select **`Assignments`** ... **`Assignment 1`**.

There are 10 questions in Asst1 (and 15 in Asst2).
You may attempt each question up to 10 times.
This assignment contributes approximately 8% to your final grade for Discrete Mathematics.

Deadline: 5pm, Monday 3 October 2022.
Deadline (Asst 2): 5pm, Friday 14 October 2022.

MA284
Week 4: Algebraic and Combinatorial Proofs

Start of ...

PART 1: A short summary

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as "} n \text{ choose } k \text{"}$$

1. $\binom{n}{k} = |\mathbf{B}^n_k|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ **is the number of ways to select k objects from a total of n objects.**

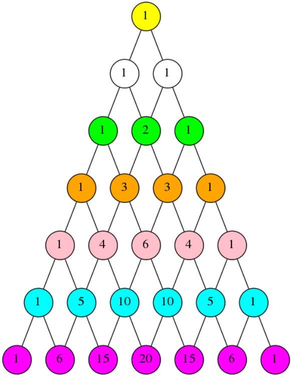
There is a formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can also calculate binomial coefficients using Pascal's identity.

Pascal's Identity: a recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

Permutations of k objects from n

The number of permutations of k objects out of n, $P(n, k)$, is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

MA284
Week 4: Algebraic and Combinatorial Proofs

END OF PART 1

MA284
Week 4: Algebraic and Combinatorial Proofs

Start of ...

PART 2: Pascal's Triangle (again)

At the end of Week 3, we "proved" that

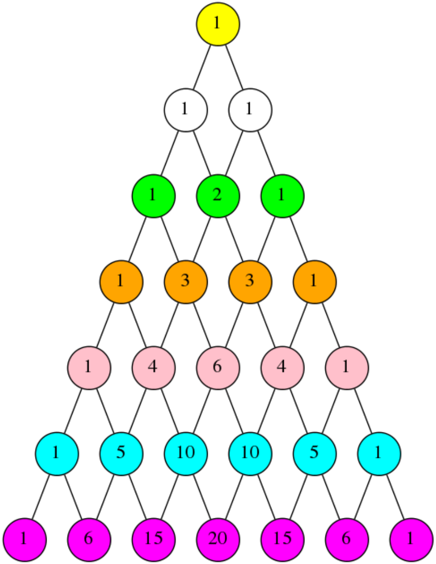
$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We did this by counting $P(n, k)$ in two different ways.

This is a classic example of a *Combinatorial Proof*, where we establish a formula by counting something in 2 different ways.

For much of this week, we will study this style of proof. See also Section 1.4 of the text-book.

But first, we will form some conjectures, using **Pascal's Triangle**.



Binomial coefficients have many important properties.
Looking at their arrangement in Pascal's Triangle, we can spot some:

(i) For all n, $\binom{n}{0} = \binom{n}{n} = 1$

(ii) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

(iii) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

(iv) $\binom{n}{k} = \binom{n}{n-k}$

MA284
Week 4: Algebraic and Combinatorial Proofs

END OF PART 2

MA284
Week 4: Algebraic and Combinatorial Proofs

Start of ...

PART 3: Algebraic and Combinatorial Proofs

Proofs

Proofs of identities involving Binomial coefficients can be classified as

- **Algebraic:** if they rely mainly on the formula for binomial coefficients.
- **Combinatorial:** if they involve counting a set in two different ways.

For our first example, we will give two proofs of the following fact:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Algebraic proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Combinatorial Proofs

Proofs of identities involving **binomial coefficients** can be classified as either

- **Algebraic:** if they rely mainly on the formula for binomial coefficients; or
- **Combinatorial:** if they involve counting a set in two different ways.

Example

Give two proofs of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

First, we check:

Algebraic proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Combinatorial proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

MA284
Week 4: Algebraic and Combinatorial Proofs

END OF PART 3

MA284
Week 4: Algebraic and Combinatorial Proofs

Start of ...

PART 4: How combinatorial proofs work

Which are better: algebraic or combinatorial proofs?

When we first study discrete mathematics, *algebraic* proofs make seem easiest: they reply only on using some standard formulae, and don't require any deeper insight. Also, they are more "familiar".

However,

- Often algebraic proofs are quite tricky;
- Usually, algebraic proofs give no insight as to why a fact is true.

Example (MA284 - Semester 1 exam, 2016/2017)

Give a combinatorial proof of the following fact

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

We wish to show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

What is a "combinatorial proof" really?

1. These proofs involve finding two different ways to answer the same counting question.
2. Then we explain why the answer to the problem posed one way is A
3. Next we explain why the answer to the problem posed the other way is B.
4. Since A and B are answers to the same question, we have shown it must be that $A = B$.

Example

Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 1:

Example

Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 2:

MA284
Week 4: Algebraic and Combinatorial Proofs

END OF PART 4

Unless indicated otherwise, these questions identical to, or variants on, Sections 1.4, 1.5 and 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. Put the following numbers in increasing order.

(a) The number of subsets of the set $\{a, b, c, d, e, g, h, i\}$.
(b) $\binom{10}{5}$
(c) $\binom{12}{3}$.
(d) $\binom{12}{3}$.
(e) 5!
(f) $P(7, 4)$
(g) $P(8, 5)$

Q2. Compute $\binom{7}{3}$ using Pascal's Identity. Check you got the right answer by also doing this using the factorial formula.

Q3. Write out **all** permutations of the letters `A`, `B`, `C`, and `D` that use all four letters. Verify you get 24.
Now write out **all** permutations of the 4 letters `A`, `B`, `C`, and `C` (i.e., C is repeated). How many do you get?

Q4. Give a combinatorial proof for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

Q5. Give an algebraic proof, using induction, for the identity
$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

Q6. Give a combinatorial proof of the fact that $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy$

Q7. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

Q8. Consider the bit strings in $\mathbf{B}^6_2$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 01?
(b) How many of those bit strings start with 001?
(c) Are there any other strings we have not counted yet? Which ones, and how many are there?
(d) How many bit strings are there total in $\mathbf{B}^6_2$?
(e) What binomial identity have you just given a combinatorial proof for?

Q9. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}.$$

Q10. (MA284 – Semester 1 exam, 2017/2018) combinatorial argument, or otherwise, prove the following statement.

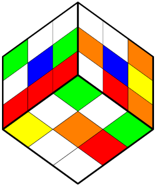
$$\binom{n}{5} = \binom{2}{2}\binom{n-3}{2} + \binom{3}{2}\binom{n-4}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n-3}{2}\binom{2}{2}.$$

MA284 : Discrete Mathematics

Week 5: Stars and Bars

Dr Kevin Jennings

5 and 7 October, 2022

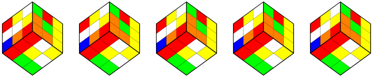


ASSIGNMENT 1 should have been closed
ASSIGNMENT 2 is due Oct 18 (extended from Oct 14)
ASSIGNMENT 3 will open soon

MA284
Week 5: Stars and Bars

Start of ...

PART 1: Stars and Bars



Think about the following question during this lecture...

Suppose you have some number of identical Rubik's cubes to distribute to your friends. Find the number of different ways you can distribute the cubes...

1. if you have 3 cubes to give to 2 people.
2. if you have 4 cubes to give to 2 people.
3. if you have 5 cubes to give to 2 people.
4. if you have 3 cubes to give to 3 people.
5. if you have 4 cubes to give to 3 people.
6. if you have 5 cubes to give to 3 people.

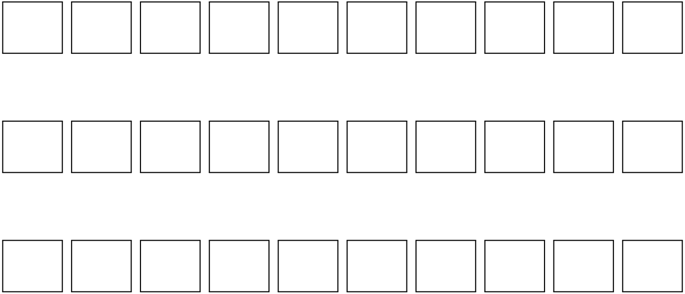
Make a conjecture about how many different ways you could distribute 7 cubes to 4 people. Explain.

What if each person were required to get *at least one* cube? How would your answers change?

Every day you give some apples to your lecturers.
Today you have **7** apples.
How many ways can you give them to **4** lecturers you have today?

Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

One can represent any solution by filling out 10 boxes with 7 stars and 3 bars.
Examples:



Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

- *Every solution can be represented by 10 boxes, each with a star or a bar.*
- There are 7 stars and 3 bars in total.
- We can choose any 3 of the 10 boxes in which to place the bars, and then put the stars in the rest.
- **So we have $\binom{10}{3}$ choices for where to put the bars.**

Definition (Multiset)

A ***multiset*** is a set of objects, where each object can appear more than once. As with an ordinary set, order does not matter.

Examples:

How many *multisets* of size 4 can you form using numbers $\{1, 2, 3, 4, 5\}$?

How many *multisets* of size n can you form using the numbers $\{1, 2, 3, \ldots, k\}$?

Example

1. In how many ways can one distribute **ten** €1 coins to four students?
2. In how many ways can one distribute **ten** €1 coins to four students so that each student receives at least €1?

MA284
Week 5: Stars and Bars

END OF PART 1

MA284
Week 5: Stars and Bars

Start of ...

PART 2: Problems with NNI solutions

In Part 1, we had the following question

How many ways can you share **n** apples among your **k** lecturers?

- This is the same as finding the number of ways we can arrange n apples (stars), divided into k groups, separated by $k-1$ bars.
- Any way can be written with $n+k-1$ symbols (n stars and $k-1$ bars): we just have to choose where to put the $k-1$ bars. This can be done in $\binom{n+k-1}{k-1} = \binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$ ways.
- Each solution can be thought of as a *multiset*: a set of objects, where each object can appear more than once.
- And each can be framed as a solution to the **non-negative integer problem:**

$$x_1 + x_2 + \cdots + x_k = n.$$

All the examples we have looked at so far this week are examples of a broader class of **non-negative integer (NNI) problems**. When we calculate the number of ways of giving 7 apples to 4 lecturers, we are computing the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 7.$$

A non-negative integer (NNI) problem

How many non-negative integer solutions are there to the problem

$$x_1 + x_2 + \cdots + x_k = n?$$

This is the same as...

How many ways are there to distribute n identical objects among k individuals.

The answer is $\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}$

MA284
Week 5: Stars and Bars

END OF PART 2

MA284
Week 5: Stars and Bars

Start of ...

PART 3: Inequalities

Example (Part 1: Equality)

What are the non-negative integer solutions to

$$x_1 + x_2 + x_3 = 3?$$

Here $n = 3$ and $k = 3$. So we know there are $\binom{5}{2} = 10$ solutions.

They are:

Example (Part 2: Inequality)

How many non-negative integer solutions are there to

$$x_1 + x_2 \leq 3,$$

and what are they?

Example (Part 3: Strict inequality)

How many non-negative integer solutions are there to

$$x_1 + x_2 < 4,$$

and what are they?

In fact, each of the following 3 equations have the same non-negative integer solutions (and, so, same number of solutions):

(1) $x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n$.

(2) $x_1 + x_2 + x_3 + \cdots + x_k \leq n$,
and

(3) $x_1 + x_2 + x_3 + \cdots + x_k < n + 1$,

WHY?

Example

(i) How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

(ii) How many non-negative integer solutions are there of the inequality

$$x_1 + x_2 + x_3 \leq 8$$

Example

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if

1. $x_1 \geq 3$;
2. $x_1 \geq 3$ and $x_2 \geq 3$;
3. Each $x_i \geq 3$

These problems are a little more complicated than the ones we just did.

Example (Upper bounds example 1)

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if:

1. Each $x_i \leq 2$.

Example (Upper bounds example 2)

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if:

2. Each $x_i \leq 3$.

Example (Upper bounds example 2)

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if:

3. Each $x_i \leq 4$.

MA284
Week 5: Stars and Bars

END OF PART 3

MA284
Week 5: Stars and Bars

Start of ...

PART 4: Advanced Counting Using PIE

Recall that

- $|X|$ denotes the number of elements in the set X.
- $X \cup Y$ (the *union of X and Y*) is the set of all elements that belong to **either** X or Y.
- $A \cap B$ (the *intersection of X and Y*) is the set of all elements that belong to *both* X and Y.

The **Principle of Inclusion/Exclusion (PIE)** for two sets, A and B, is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets, A, B and C, the PIE is

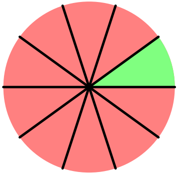
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$$|A \cup B \cup C \cup D| = \text{(the sum of the sizes of each single set)}$$
$$- \text{(the sum of the sizes of each } \textbf{intersection} \text{ of 2 sets)}$$
$$+ \text{(the sum of the sizes of each } \textbf{intersection} \text{ of 3 sets)}$$
$$- \text{(the sum of the sizes of } \textbf{intersection} \text{ of all 4 sets)}$$

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



Not all such problems have easy solution solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...}$$

1. There are no restrictions (other than each x_i being an *nni*).
2. $0 \leq x_i \leq 3$ for each i.

... continued... *2. How many non-negative integer solutions are there to*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...} 0 \leq x_i \leq 3 \text{ for each } i.$$

Unless indicated otherwise, these questions identical to, or variants on, problems in Section 1.5 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. A *multiset* is a collection of objects, just like a set, but can contain an object more than once (the order of the elements still doesn't matter). For example, $\{1, 1, 2, 5, 5, 7\}$ is a multiset of size 6.

(a) How many *sets* of size 5 can be made using the 10 digits: 0, 1, . . . 9?

(b) How many *multi*sets of size 5 can be made using the 10 digits: 0, 1, . . . 9?

Q2. Each of the counting problems below can be solved with stars and bars. For each, say what outcome the diagram $***|*||**|$ represents, if there are the correct number of stars and bars for the problem. Otherwise, say why the diagram does not represent any outcome, and what a correct diagram would look like.

(a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?

(b) How many ways can you distribute 5 identical lollipops to 6 kids?

(c) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$.

Q3. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"

(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q4. (MA284, Semester 1 Exam, 2017/2018) How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

Q5. (MA284, Semester 1 Exam, 2017/2018)

(a) How many non-negative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 < 11,$$

if there are no restrictions?

(b) How many solutions are there to the above problem if $x_2 \geq 3$?

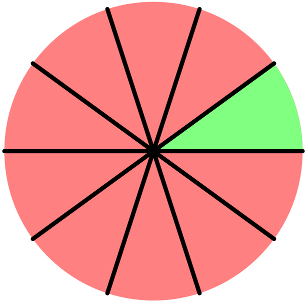
(c) How many solutions are there if each $x_i \leq 4$?

MA204/MA284 : Discrete Mathematics

Week 6: Advanced PIE, Derangements, and Counting Functions

Dr Kevin Jennings

12 and 14 October, 2022



See also §1.6 of Levin's *Discrete Mathematics: an open introduction*. Some slides are based on ones by Dr Angela Carnevale (and Dr Niall Madden of course).

Assignment 2 is now open, with a deadline of *5pm next Tuesday: 18 October*. Like Assignment 1, you must access it through Blackboard.

Assignment 3 opens very soon A practice exercise sheet pdf is already available.

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

Start of ...

PART 1: Part 1: Advanced Counting Using PIE

Recall that

- $|X|$ denotes the number of elements in the set X.
- $X \cup Y$ (the *union of X and Y*) is the set of all elements that belong to **either** X or Y.
- $X \cap Y$ (the *intersection of X and Y*) is the set of all elements that belong to *both* X and Y.

The **Principle of Inclusion/Exclusion (PIE)** for two sets, A and B, is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The **Principle of Inclusion/Exclusion (PIE)** for three sets, A, B and C, is

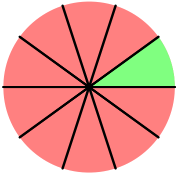
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$$\begin{aligned}|A \cup B \cup C \cup D| = &\text{(the sum of the sizes of each single set)}\\ &- \text{(the sum of the sizes of each } \textbf{intersection} \text{ of 2 sets)}\\ &+ \text{(the sum of the sizes of each } \textbf{intersection} \text{ of 3 sets)}\\ &- \text{(the sum of the sizes of } \textbf{intersection} \text{ of all 4 sets)}\end{aligned}$$

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



But it is instructive, if a little tedious, to use the PIE to answer this. (See http://discrete.openmathbooks.org/dmoi3/sec_advPIE.html for a more detailed solution):

$$\binom{13}{3} - \binom{4}{1}\binom{10}{3} + \binom{4}{2}\binom{7}{3} - \binom{4}{3}\binom{4}{3} + \binom{4}{4}\binom{1}{3} = 286 - 480 + 210 - 16 = 0$$

Not all such problems have easy solution solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...}$$

1. There are no restrictions (other than each x_i being an *nni*).
2. $0 \leq x_i \leq 3$ for each i.

... continued... *2. How many non-negative integer solutions are there to*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...} 0 \leq x_i \leq 3 \text{ for each } i.$$

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

END OF PART 1

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

Start of ...

PART 2: Derangements

For **Assignment 2** of AM842 (*Indiscreet Mathematics*), students work together in groups of **4**. The group is given a score, which they divide up, according to the amount of work each did, to get their individual scores.
Aoife, Brian, Conor and Dana worked together, and got a score of **10**. They decided it should be divided as:

Aoife: 1   Brian: 2   Conor: 3   Dana: 4.

They informed their lecturer of this, and he tried to enter these on Blackboard. But he is not very good with computers, and got ALL the scores wrong!
How many ways could this happen?

Recall: PERMUTATION

A **permutation** of a collection of objects is a re-ordering of it.
There are $n!$ permutations of a set with n elements.

DEGRANGEMENTS

A **derangement** is a permutation where no item is left in its original place.

Example:

The study of **derangements** dates back to at least 1708. The old French card game called *Rencontres* was a game of chance for two players, A and B:

- The players begin with a shuffled, full deck of 52 cards each.
- Each would take turns placing random cards on the table.
- If any of the cards matched, player A would win.
- If none of the cards matched, player B would win.

In 1708, Pierre de Montort (1678–1719) posed the problem: what is the probability that there would be no matches?

If we let D_{52} be the number of *derangements* of 52 cards then the solution is $D_{52}/52!$.

Let D_n be the number of *derangements* of n objects. First we will work out formulae for D_1, D_2, D_3, and D_4.

In general, the formula for D_n, the number of derangements of n objects is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

END OF PART 2

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

Start of ...

PART 3: Counting with repetitions

Suppose we have a set of n objects which are all distinguishable.

(a) How many k-permutations are there (with no repetition)?

(b) How many k-combinations are there (with no repetition)?

But what happens when *some* of the elements of the set are *indistinguishable*?

How many "words" can we make from the following sets of letters?

(i) $\{\texttt{M},\texttt{A},\texttt{Y},\texttt{O}\}$

(ii) $\{\texttt{C},\texttt{L},\texttt{A},\texttt{R},\texttt{E}\}$

(iii) $\{\texttt{G},\texttt{A},\texttt{L},\texttt{W},\texttt{A},\texttt{Y}\}$

(iv) $\{\texttt{R},\texttt{O},\texttt{S},\texttt{C},\texttt{O},\texttt{M},\texttt{M},\texttt{O},\texttt{N}\}$

Let's consider the last example carefully: how many "words" can we make from letters in the set $\{\texttt{C},\texttt{M},\texttt{M},\texttt{N},\texttt{O},\texttt{O},\texttt{O},\texttt{R},\texttt{S}\}$?

If somehow the three O's were all distinguishable, and the two M's were distinguishable, the answer would be $9!$.

But, since we can't distinguish identical letters,

- Let's choose which of the 9 positions we place the three O's. This can be done in $\binom{9}{3}$ ways.
- Now let's choose which of the remaining 6 positions we place the two M's. This can be done in $\binom{6}{2}$ ways.
- Now let's choose where to place the remaining 4 letters. This can be done in $4!$ ways.

By the Multiplicative Principle, the answer is

$$\binom{9}{3}\binom{6}{2}4! = \frac{9!}{3!6!}\frac{6!}{2!4!}4! = \frac{9!}{3!2!}$$

Multinomial coefficient

The number of different permutations of n objects, where there are n_1 indistinguishable objects of Type 1, n_2 indistinguishable objects of Type 2, ..., and n_k indistinguishable objects of Type k, is

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}$$

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name `WOLLONGONG`.

(ii) How many of these arrangements start with the three `O`'s;

(iii) How many contain the two `G`'s consecutively;

(iv) How many *do not* contain the two `G`'s consecutively?

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

END OF PART 3

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

Start of ...

PART 4: Counting Functions

Recall the $f : A \to B$ is a *function* that maps every element of the set A onto some element of set B. (We call A the "domain", and B the "codomain".)
Each element of A gets mapped to exactly one element of B.

If $f(a) = b$ where $a \in A$ and $b \in B$, we say that "the image of a is b".
Or, equivalently, "b is the image of a".

Examples:

When every element of B is the image of some element of A, we say that the function is **surjective** (also called "onto").

Examples:

When no two elements of A have the same image in B, we say that the function is **injective** (also called "one-to-one").

Examples:

Bijection

The function $f : A \to B$ is a **bijection** if it is both *surjective* and *injective*. Then f defines a *one-to-one correspondence* between A and B.

Counting functions

Let A and B be finite sets. How many functions $f\colon A \to B$ are there?

We can use the Multiplicative Principle to deduce:

There are in total $|B|^{|A|}$ functions from A to B.

Counting Bijective Functions (Example 1.3.2 of the textbook)

How many functions $f : \{1, 2, 3, 4, 5, 6, 7, 8\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are bijective?

Remember what it means for a function to be bijective: **each element in the codomain** must be the image of **exactly one element of the domain**. We could write one of these bijections as

What we are really doing is just rearranging the elements of the codomain, so we are defining a **permutation** of 8 elements.

The answer to our question is therefore 8!.

More in general, there are $n!$ bijections of the set $\{1, 2 \ldots, n\}$ onto itself.

Counting Injective Functions (Example 1.3.2 of the textbook)

How many functions $f\colon \{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ are injective?

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is

$P(8,3) = 8 \cdot 7 \cdot 6.$

Similarly, we can see a k-permutation of $\{1,2,3,\dots,n\}$ as an injective function from $\{1,2,\dots,k\}$ to $\{1,2,3,\dots,n\}$. In general, the number of such injections is $P(n,k)$.

Finally, **derangements** can be interpreted as bijections from a set onto itself and **without fixed points**.

Counting functions without fixed points (see also Section 1.6 of the textbook)

How many **bijective** functions $f\colon \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ are there such that $f(x) \neq x$ for all $x \in \{1,2,3,4,5\}$?

Using our formula

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44.$$

MA284
Week 6: Advanced PIE, Derangements, and Counting Functions

END OF PART 4

Most of these questions are based on exercises in Section 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"

(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q2. (MA284, Semester 1 Exam, 2017/2018) How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

Q3. (MA284, Semester 1 Exam, 2017/2018)

(a) How many non-negative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 < 11,$$

if there are no restrictions?

(b) How many solutions are there to the above problem if $x_2 \geq 3$?

(c) How many solutions are there if each $x_i \leq 4$?

Q4. The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

(a) No present is allowed to end up with its original label? Explain what each term in your answer represents.
(b) Exactly 2 presents keep their original labels? Explain.
(c) Exactly 5 presents keep their original labels? Explain.

Q5. (MA284 Semester 1 Exam, 2018/2019) Let D_n be the number of derangements of n objects. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Q6. (MA284 Semester 1 Exam, 2018/2019) Five students' phones were confiscated on their way into an exam. How many ways can their phones be returned to them so that no one gets their own phone back? How many ways can their phones be returned to them so exactly two people get their own phones back?

Q7. (MA284 Semester 1 Exam, 2015/2016) Give a formula for the number distinct permutations (arrangements) of all the letters in the word `BALLYGOBACKWARDS`.
How many of these begin with an "L"?
How many have all the vowels together?
How many have all the letters in alphabetical order?

Q8. Consider functions $f : \{1, 2, 3, 4\} \to \{a, b, c, d, e, f\}$. How many functions have the property that $f(1) \neq a$ or $f(2) \neq b$, or both?

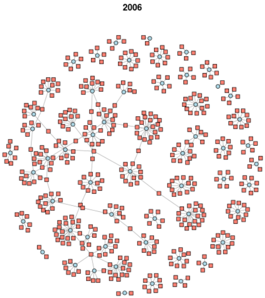
Q9. Consider sets A and B with $|A| = 10$ and $|B| = 5$. How many functions $f : A \to B$ are *surjective*? [*Hint: the answer is* $5^{10} - 5 \times 4^{10} + 10 \times 3^{10} - 10 \times 2^{10} - 5$. *But why?*]

MA284 : Discrete Mathematics

Week 7: Introduction to Graph Theory

Dr Kevin Jennings

19 and 21 October, 2022



See also §1.6, §4.0 and §4.1 of Levin's *Discrete Mathematics: an open introduction*. Some slides are based on ones by Angela Carnevale, all other good stuff due to Niall Madden, mistakes Kevin's.

- **Assignment 1** is closed. Your grade is available on Blackboard.
- **Assignment 2 is open until Thursday afternoon**. You need to access it through Blackboard.
- **Assignment 3** is now open, with a deadline of *5pm, Thursday: 4 November*

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name WOLLONGONG.

(ii) How many of these arrangements start with the three O's;

(iii) How many contain the two G's consecutively;

(iv) How many *do not* contain the two G's consecutively?

MA284
Week 7: Introduction to Graph Theory

Start of ...

PART 1: Counting Functions

(This is actually left over from last week, and not really related to the main topic of the week: Graph Theory)

Recall the $f : A \to B$ is a *function* that maps every element of the set A onto some element of set B. (We call A the "domain", and B the "codomain".)
Each element of A gets mapped to exactly one element of B.

If $f(a) = b$ where $a \in A$ and $b \in B$, we say that "the image of a is b".
Or, equivalently, "b is the image of a".

Examples:

When every element of B is the image of some element of A, we say that the function is *SURJECTIVE* (also called "onto").

Examples:

When no two elements of A have the same image in B, we say that the function is *INJECTIVE* (also called "one-to-one").

Examples:

Bijection

The function $f : A \rightarrow B$ is a **BIJECTION** if it is both *surjective* and *injective*. Then f defines a *one-to-one correspondence* between A and B.

Counting functions

Let A and B be finite sets. How many functions $f: A \to B$ are there?

We can use the Multiplicative Principle to deduce:

There are in total $|B|^{|A|}$ functions from A to B.

Counting Bijective Functions (Example 1.3.2 of the textbook)

How many functions $f : \{1, 2, 3, 4, 5, 6, 7, 8\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are bijective?

Remember what it means for a function to be bijective: **each element in the codomain** must be the image of **exactly one element of the domain**. We could write one of these bijections as

What we are really doing is just rearranging the elements of the codomain, so we are defining a **permutation** of 8 elements.

The answer to our question is therefore $8!$.

More generally, there are $n!$ bijections of the set $\{1, 2 \ldots, n\}$ onto itself.

Counting Injective Functions (Example 1.3.2 of the textbook)

How many functions $f\colon \{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ are injective?

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is

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Finally, **derangements** can be interpreted as bijections from a set onto itself and **without fixed points**.

Counting functions without fixed points (see also Section 1.6 of the textbook)

How many **bijective** functions $f\colon \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ are there such that $f(x) \neq x$ for all $x \in \{1,2,3,4,5\}$?

Using our formula

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44.$$

MA284
Week 7: Introduction to Graph Theory

END OF PART 1

MA284
Week 7: Introduction to Graph Theory

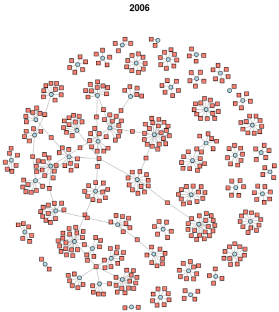
Start of ...

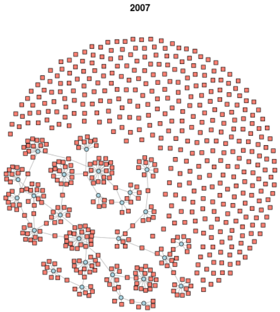
PART 2: Graph Theory

An introduction...

Graph Theory is a branch of mathematics that is several hundred years old. Many of its discoveries were motivated by practical problems, such as determining the smallest number of colours needed to colour a map.

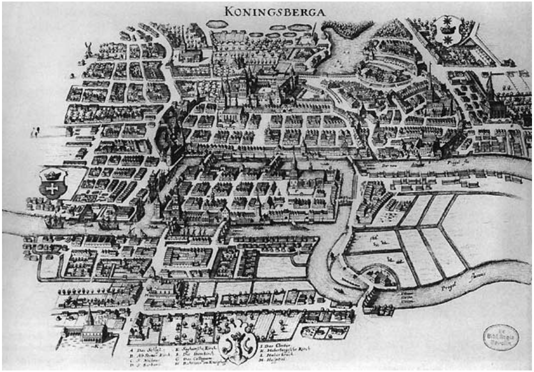
However, it remains one of the most important and exciting areas of modern mathematics, as a bed-rock of data sciences and network theory.





Graph Theory is unusual in that its beginnings can be traced to a precise date.

Königsberg in Prussia (now Kaliningrad, Russia) had seven bridges. Is it possible to walk through the town in such a way that you cross each bridge once and only once?



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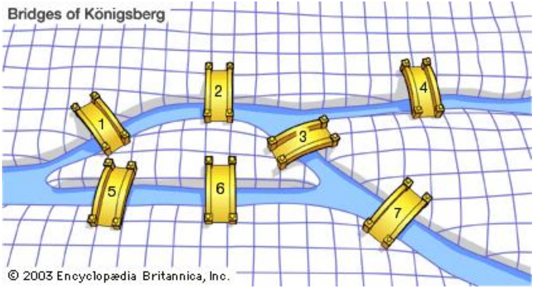

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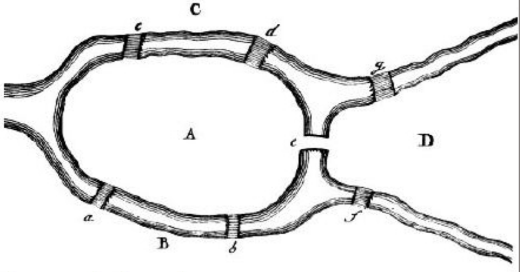
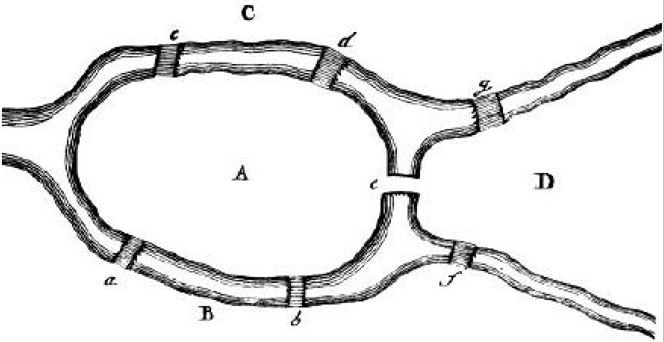


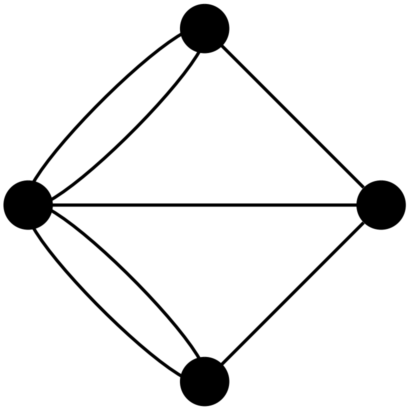
The seven bridges of Königsberg

Is it possible to walk through the town in such a way that you cross each bridge once and only once?



The seven bridges of Königsberg

Here is another way of stating the same problem. Consider the following picture, which shows 4 dots connected by some lines.



Is it possible to trace over each line once and only once (without lifting up your pencil)? You must start and end on one of the dots.

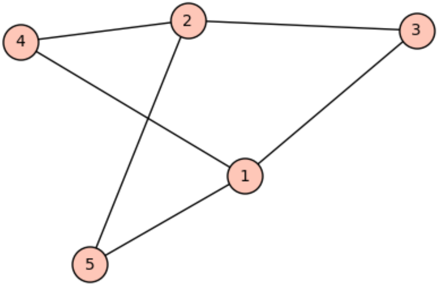
Graph

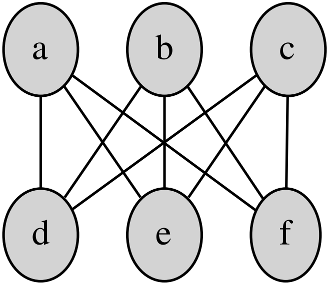
A *GRAPH* is a collection of

- "vertices" (or "nodes"), which are the "dots" in the above diagram.
- "edges" joining pair of vertices.

If the graph is called G (say), we often define it in terms of its *edge set*, E, and *vertex set*, V, as

$$G = (V, E).$$

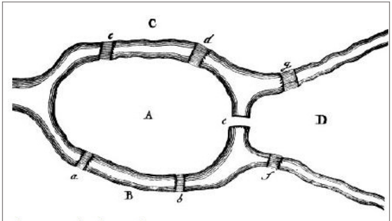




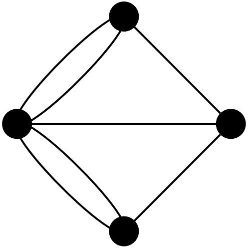
If two vertices are connected by an edge, we say they are *adjacent*.

Graphs are used to represent collections of objects where there is a special relationship between certain pairs of objects.

For example, in the Königsberg problem, the land-masses are vertices, and the edges are bridges.



The seven bridges of Königsberg



(Example 4.0.1 of the text-book)

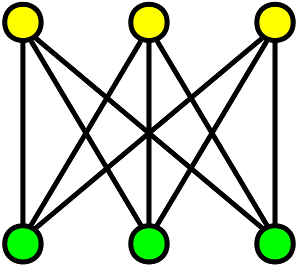
Aoife, Brian, Conor, David and Edel are students in an *Indiscrete Mathematics* module.

- Aoife and Conor worked together on their assignment.
- Brian and David also worked together on their assignment.
- Edel helped everyone with their assignments.

Represent this situation with a graph.

The Three Utilities Problem; also Eg 4.0.2 in text-book

We must make Water, Power and Gas connections to three houses.
Is it possible to do this without the conduits crossing?



MA284
Week 7: Introduction to Graph Theory

END OF PART 2

MA284
Week 7: Introduction to Graph Theory

Start of …

PART 3: Graph Theory - The Basics

Key terms and notation

Definition (ORDER)

The order a graph $G = (V, E)$ is the size of its vertex set, $|V|$.

Let $G = (V, E)$, with

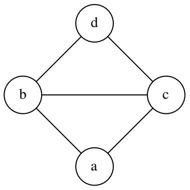
$$V = \{a, b, c, d\}, \qquad E = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\big\}$$

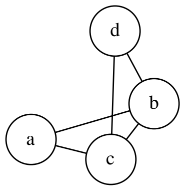
What is the order of G? Sketch G.

Two graphs are *EQUAL* if the have exactly the same Edge and Vertex sets. That is *it is not important how we draw them*, how where we position the vertices, the length of the edges, etc.

Example (Section 4.1 of text-book)

Show that the two graphs given below are *equal*





Isomorphism

An *ISOMORPHISM* between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a *bijection* $f : V_1 \to V_2$ between the vertices in the graph such that, if $\{a, b\}$ is an edge in G_1, then $\{f(a), f(b)\}$ is an edge in G_2.
Two graphs are *ISOMORPHIC* if there is an isomorphism between them. In that case, we write $G_1 \cong G_2$.

Example (Example 4.1.1 of text-book)

Show that the graphs

$$G_1 = \{V_1, E_1\}, \text{ where } V_1 = \{a, b, c\} \text{ and } E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\};$$

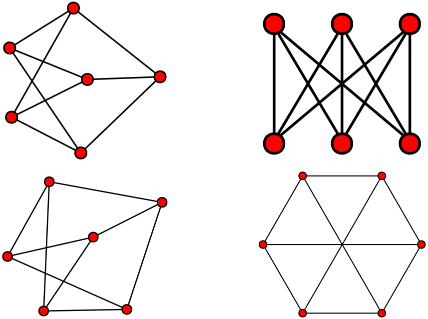
$$G_2 = \{V_2, E_2\} \text{ where } V_2 = \{u, v, w\}, \text{ and } E_2 = \{\{u, v\}, \{u, w\}, \{v, w\}\}$$

are not *equal* but are *isomorphic*.

Example (Example 4.1.3 from text-book)

Decide whether the graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are equal or isomorphic, where
$V_1 = \{a, b, c, d\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\}$ and
$V_2 = \{a, b, c, d\}$, $E_2 = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$

When we give a graph without labeling the vertices, we are really talking about *all* graphs that are **isomorphic** to the one we have just drawn. For example, when we draw the following graph, we mean it to represent all those graphs that are isomorphic to the *Water-Power-Gas* graph.



Other than the Königsberg Bridges example, all the graphs we have looked at so far

1. have no *loops* (i.e., no edge from a vertex to itself).
2. have no repeated edges (i.e., there is at most one edge between each pair of vertices).

Such graphs are called *SIMPLE* graphs. But because they are the most common, unless we say otherwise, when we say "graph" we mean "simple graph".

If a graph does have repeated edges, like in the Königsberg example, we call it a *MULTIGRAPH*. Then the list of edges is not a set, since some elements are repeated: it is a multiset (see Week 5).

MA284
Week 7: Introduction to Graph Theory

END OF PART 3

MA284
Week 7: Introduction to Graph Theory

Start of ...

PART 4: Walks, paths, cycles and circuits

Definition (WALK, TRAIL, PATH)

A **WALK** is sequence of vertices such that consecutive vertices are adjacent.
A **TRAIL** is walk in which no edge is repeated.
A **PATH** is a trail in which no vertex is repeated, except possibly the first and last.

Example:

We can also describe a path by the edge sequence. This can be useful, since the **LENGTH** of the path is the number of *edges* in the sequence.

And, since there can be more than one, the **SHORTEST PATH** is particularly important.

Example:

Cycles and Circuits

There are two special types of **path** that we will study later in detail:

- Cycle: A path that begins and ends at that same vertex, but no other **vertex** is repeated;
- Circuit: A path that begins and ends at that same vertex, and no **edge** is repeated;

These questions are based on exercises in Sections 1.6 and 4.1 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. Consider functions $f : \{1, 2, 3, 4\} \to \{a, b, c, d, e, f\}$. How many functions have the property that $f(1) \neq a$ or $f(2) \neq b$, or both?

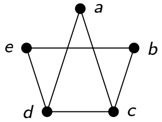
Q2. Consider sets A and B with $|A| = 10$ and $|B| = 5$. How many functions $f : A \to B$ are *surjective*? [*Hint: the answer is* $5^{10} - 5 \times 4^{10} + 10 \times 3^{10} - 10 \times 2^{10} - 5$. *But why?*]

Q3. (Exercise 4.1.1 from text-book) If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

Q4. (Exercise 4.0.2 of text-book *and* MA284 Semester 1 Exam, 2015/2016) Among a group of five people, is it possible for everyone to be friends with exactly two of the other people in the group?
Is it possible for everyone to be friends with exactly three of the other people in the group? Explain your answers carefully.

Q5. Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.

Graph 1: $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, e\}\}$.

Graph 2: 

Q6. (MA284, Semester 1 Exam, 2016/2017) For each of the following pairs of graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, determine if they are isomorphic. If they are, give an isomorphism between them. If not, explain why.

(a) $V_1 = \{a, b, c, d\}$, $E_1 = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\Big\}$ and
$V_2 = \{w, x, y, z\}$, $E_2 = \Big\{\{y, x\}, \{x, z\}, \{z, w\}, \{z, y\}\Big\}$.

(b) $V_1 = \{a, b, c\}$, $E_1 = \Big\{\{a, b\}, \{b, c\}, \{a, c\}\Big\}$ and
$V_2 = \{w, x, y, z\}$, $E_2 = \Big\{\{w, z\}, \{z, y\}, \{w, x\}\Big\}$.

(c) $V_1 = \{a, b, c, d, e\}$, $E_1 = \Big\{\{a, c\}, \{a, e\}, \{b, c\}, \{b, d\}, \{e, c\}, \{d, e\}\Big\}$ and
$V_2 = \{v, w, x, y, z\}$, $E_2 = \Big\{\{v, x\}, \{x, y\}, \{y, z\}, \{z, v\}, \{z, x\}, \{x, w\}\Big\}$.

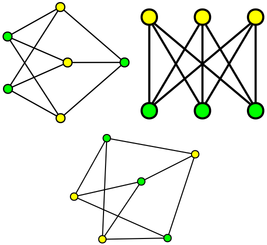
MA284 : Discrete Mathematics

Week 8: Definitions, and Planar Graphs

Dr Kevin Jennings

26 and 28 October, 2022

1. Part 1: Definitions
   - Review
   - [from Wk7: Walks, paths, cycles and circuits]
   - Connected graphs
   - Vertex degree
2. Part 2: Types of Graphs
   - Complete graphs
   - Bipartite graphs
   - Subgraphs
   - Named graphs
3. Part 3: Planar graphs
   - Faces, edges and vertices
4. Part 4: Euler's formula
5. Part 5: Non-planar graphs
   - K_5
   - $K_{3,3}$
   - Every other non-planar graph
6. Exercises



See also Sections 4.1 and 4.3 of Levin's *Discrete Mathematics.*

Some slides are based on ones by Dr Angela Carnevale, most are by Dr Niall Madden, errors are Kevin's.

MA284
Week 8: Definitions, and Planar Graphs

Start of ...

PART 1: Definitions

An "information dump" on terminology we'll use over the next number of weeks.

- A *GRAPH* is a collection of
  - "vertices" (or "nodes"), which are the "dots" in the above diagram.
  - "edges" joining pair of vertices.
- A graph is defined in terms of its *edge set* and *vertex set*. That, the graph G with vertex set V and edge set E is written as $G = (V, E)$.
- The **ORDER** of a graph is the number of vertices it has. That is, the order of $G = (V, E)$ is $|V|$.
- If two vertices are connected by an edge, we say they are *adjacent*.
- Two graphs are *EQUAL* if the have exactly the same Edge and Vertex sets.
- An *ISOMORPHISM* between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a *bijection* $f : V_1 \to V_2$ between the vertices in the graph such that, if $\{a, b\}$ is an edge in G_1, then $\{f(a), f(b)\}$ is an edge in G_2.
- Two graphs are *ISOMORPHIC* if there is an isomorphism between them. In that case, we write $G_1 \cong G_2$.
- A **WALK** is sequence of vertices such that consecutive vertices are adjacent.
- A ***trail*** is a walk in which no edge is repeated is called a trail.
- A ***path*** is a trail in which no vertex is repeated, except possibly the first and last.

Definition (WALK, TRAIL, PATH)

A **WALK** is sequence of vertices such that consecutive vertices are adjacent.
A **TRAIL** is walk in which no edge is repeated.
A **PATH** is a trail in which no vertex is repeated, except possibly the first and last.

Example:

We can also describe a path by the edge sequence. This can be useful, since the **LENGTH** of the path is the number of *edges* in the sequence.

And, since there can be more than one, the **SHORTEST PATH** is particularly important.

Example:

Cycles and Circuits

There are two special types of **path** that we will study later in detail:

Cycle: A path that begins and ends at that same vertex, but no other **vertex** is repeated;

Circuit: A path that begins and ends at that same vertex, and no **edge** is repeated;

A graph is **CONNECTED** if there is a path between every pair of vertices.

Example:

The **DEGREE** of a vertex is the number of edges emanating from it. If v is a vertex, we denote its degree as $d(v)$.

If we know the degree of every vertex in the graph then we know the number of edges. This is :

Lemma (Handshaking Lemma)

In any graph, the sum of the degrees of vertices in the graph is always twice the number of edges:

$$\sum_{v \in V} d(v) = 2|E|.$$

Example (Application of the Handshaking Lemma)

Among a group of five people,

(i) is it possible for everyone to be friends with exactly two of the other people in the group?

(ii) is it possible for everyone to be friends with exactly three of the other people in the group?

MA284
Week 8: Definitions, and Planar Graphs

END OF PART 1

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PART 2: Types of Graphs

Some very important examples of graphs that have special properties

A graph is **COMPLETE** if every pair of vertices are adjacent. This family of graphs is VERY important. They are denoted K_n – the complete graph on n vertices.

If it is possible to *partition* the vertex set, V, into two disjoint sets, V_1 and V_2, such that there are no edges between any two vertices in the same set, then the graph is **BIPARTITE**.

When the bipartite graph is such that *every* vertex in V_1 is connected to *every* vertex in V_2 (and *vice versa*) the graph is a **COMPLETE BIPARTITE GRAPH**. If $|V_1| = m$, and $|V_2| = n$, we denote it $K_{m,n}$.

We say that $G_1 = (V_1, E_1)$ is a **SUBGRAPH** of $G_2 = (V_2, E_2)$ provided $V_1 \subset V_2$, and $E_1 \subset E_2$.

We say that $G_1(V_1, E_1)$ is an **INDUCED SUBGRAPH** of $G_2 = (V_2, E_2)$ provided that $V_1 \subset V_2$ and E_2 contains *all* edges of E_1 which join edges in V_1 .

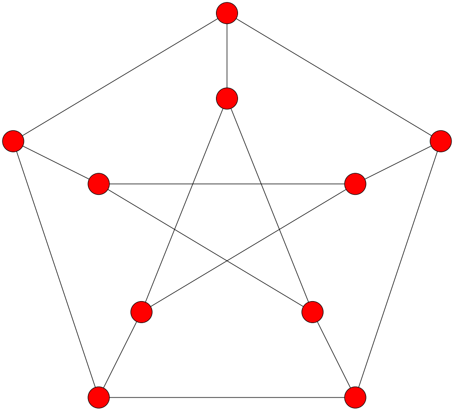
Some graphs are used more than others, and get special names. We already had

- K_n – the complete graph on n vertices.
- $K_{m,n}$ – The complete bipartite graph with sets of m and n vertices.

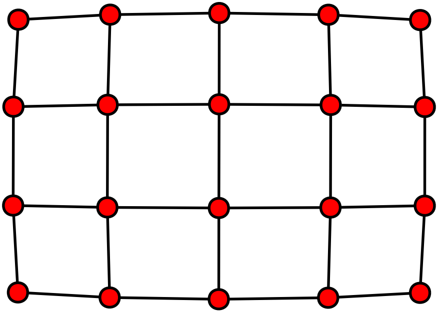
Other important ones include

- C_n – The cycle on n vertices.
- P_n – The path on n vertices.

And there are some graphs that are named after people. The most famous is the *Petersen Graph*.



Two personal favourites are the *Square Grid Graph* and *Triangular Grid Graph*.



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Week 8: Definitions, and Planar Graphs

END OF PART 2

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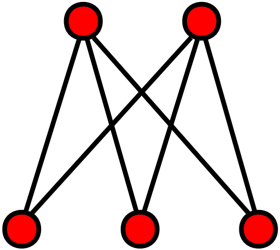
PART 3: Planar graphs

Planar graph

If you can sketch a graph so that none of its edges cross, then it is a *planar* graph.

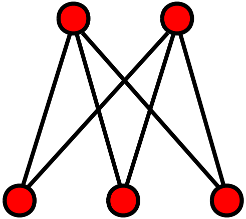
Example: The Graph $K_{2,3}$ is *planar:*



These graphs are *equal*. The sketch on the right (see annotated notes) is a *planar representation* of the graph.

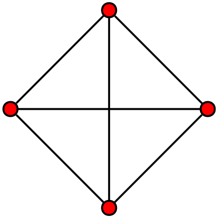
When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions.
Each region is called a *face*.

Example: the planar representation of $K_{2,3}$ has **3 faces** (because the "outside" region counts as a face).



The number of faces does not change no matter how you draw the graph, as long as no edges cross.

Example: Give a planar representation of K_4, and count how many faces it has.



More examples: Count the number of edges, faces and vertices in the cycle graphs C_3, C_4 and C_5. What about C_k?

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Week 8: Definitions, and Planar Graphs

END OF PART 3

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PART 4: Euler's formula for planar graphs

Planar graphs are very special in many ways. One of those ways is that there is a relationship between the number of faces, edges and vertices.

Presently, we'll study a famous formula relating the number of vertices, edges and faces in a *planar* graph.

First, let's try to discover it.

Example: Count the number of vertices, edges and faces of $K_{2,4}$.

Example: Count the number of vertices, edges and faces of P_2, C_3, K_4, Dickie-bow...

We have produced a list of some planar graphs and counted their vertices, edges, and faces. There is a pattern...

Euler's formula for planar graphs

For any (connected) planar graph with v vertices, e edges and f faces, we have

$$v - e + f = 2$$

Outline of proof:

(Proof continued).

Example (Application of Euler's formula)

Is it possible for a connected planar graph to have 5 vertices, 7 edges and 3 faces? Explain.

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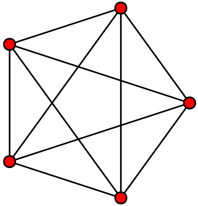
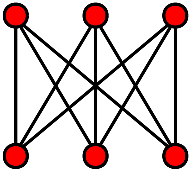
PART 5: Non-planar graphs

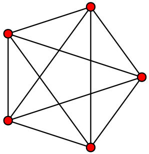
Of course, most graphs do **not** have a planar representation. We have already met two that (we think) cannot be drawn so no edges cross: K_5 and $K_{3,3}$:

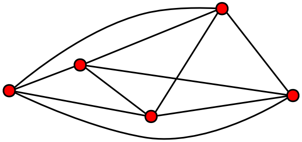




However, it takes a little work to *prove* that these are non-planar. While, through trial and error, we can convince ourselves these graphs are not planar, a proof is still required.

For this, we can use **Euler's formula for planar graphs** to *prove* they are not planar.





Theorem (Theorem 4.3.1 in textbook)

K_5 is not planar.

The proof is by *contradiction*:

Theorem ($K_{3,3}$ is not planar)

This is Theorem 4.2.2 in the text-book. Please read the proof there.

The proof for $K_{3,3}$ is somewhat similar to that for K_5, but also uses the fact that a bipartite graph has no 3-edge cycles.

This also means we have solved (negatively) the Utilities (Water-Power-Gas) problem from last week.

To understand the importance of K_5 and $K_{3,3}$, we first need the concept of *homeomorphic* graphs.

Recall that a graph G_1 is a *subgraph* of G if it can be obtained by deleting some vertices and/or edges of G.

A *SUBDIVISION* of an edge is obtained by "adding" a new vertex of degree 2 to the middle of the edge.

A *SUBDIVISION* of a graph is obtained by subdividing one or more of its edges.

Example:

Closely related: **SMOOTHING** of the pair of edges $\{a, b\}$ and $\{b, c\}$, where b is a vertex of degree 2, means to remove these two edges, and add $\{a, c\}$.

Example:

The graphs G_1 and G_2 are *HOMEOMORPHIC* if there is some subdivision of G_1 which is isomorphic to some subdivision of G_2.

Examples:

There is a *celebrated* theorem due to Kazimierz Kuratowski. *The proof is beyond what we can cover in this module. But if you are interested in Mathematics, read up in it: it really is a fascinating result.*

Theorem (Kuratowski's theorem)

A graph is planar if and only if it does not contain a subgraph that is homeomorphic to K_5 or $K_{3,3}$.

What this *really* means is that *every* non-planar graph has some smoothing that contains a copy of K_5 or $K_{3,3}$ somewhere inside it.

Example

The Petersen graph is not planar `https://upload.wikimedia.org/wikipedia/commons/0/0d/Kuratowski.gif`

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END OF PART 5

Most of these questions are taken from Levin's *Discrete Mathematics*.

Q1. Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

Q2. Try to prove that $K_{3,3}$ is non-planar using *exactly* the same reasoning as that used to prove K_5 is non-planar. What does wrong? (The purpose of this exercise is to show that noting that $K_{3,3}$ has no 3-cycles is key. Also, we want to know that K_5 and $K_{3,3}$ are non-planar for different reasons).

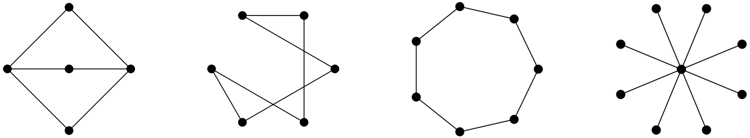
Q3. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

Q4. The graph G has 6 vertices with degrees 2,2,3,4,4,5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

Q5. Euler's formula ($v - e + f = 2$) holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has k components?

Q6. Prove that any planar graph with v vertices and e edges satisfies $e \leq 3v - 6$.

Q7. Which of the graphs below are bipartite? Justify your answers.



Q8. For which $n \geq 3$ is the graph C_n bipartite?

Q9. For each of the following, try to give two different unlabeled graphs with the given properties, or explain why doing so is impossible.

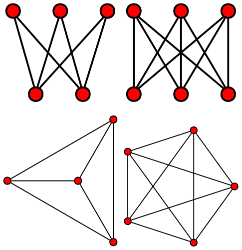
(a) Two different trees with the same number of vertices and the same number of edges. (A tree is a connected graph with no cycles).
(b) Two different graphs with 8 vertices all of degree 2.
(c) Two different graphs with 5 vertices all of degree 4.
(d) Two different graphs with 5 vertices all of degree 3.

MA284 : Discrete Mathematics

Week 9: Convex Polyhedra

Dr Kevin Jennings

2nd and 4th November, 2022



See also Section 4.3 of Levin's *Discrete Mathematics.*

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Week 9: Convex Polyhedra

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PART 1: Non-Planar graphs

Recall: Planar graph

- If you can sketch a graph so that none of its edges cross, then it is a *PLANAR* graph.
- When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a *FACE*. The "exterior" of the graph is considered a face.

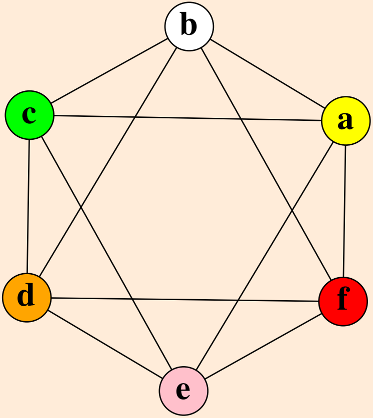
Euler's formula for planar graphs

For any (connected) planar graph with v vertices, e edges and f faces, we have

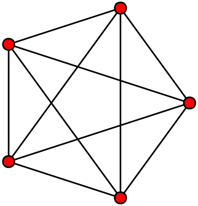
$$v - e + f = 2$$

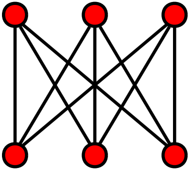
Example

Give a planar representation of the following graph, and verify that Euler's Formula Holds.



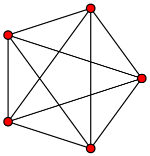
Of course, most graphs do **not** have a planar representation. We have already met two that (we think) cannot be drawn so no edges cross: K_5 and $K_{3,3}$:

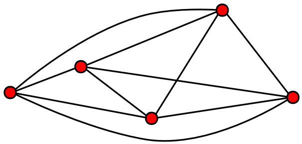




However, it takes a little work to *prove* that these are non-planar. While, through trial and error, we can convince ourselves these graphs are not planar, a proof is still required.

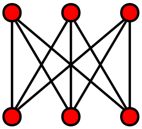
For this, we can use **Euler's formula for planar graphs** to *prove* they are not planar.

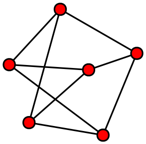




Theorem (Theorem 4.3.1 in textbook)

K_5 is not planar. (The proof is by contradiction).





Theorem ($K_{3,3}$ is not planar)

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This also means we have solved (negatively) the Utilities (Water-Power-Gas) problem from Week 7.

The understand the importance of K_5 and $K_{3,3}$, we first need the concept of *homeomorphic* graphs.

Recall that a graph G_1 is a *subgraph* of G if it can be obtained by deleting some vertices and/or edges of G.

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Closely related: **SMOOTHING** of the pair of edges $\{a, b\}$ and $\{b, c\}$, where b is a vertex of degree 2, means to remove these two edges, and add $\{a, c\}$.

Example:

The graphs G_1 and G_2 are *HOMEOMORPHIC* if there is some subdivision of G_1 is isomorphic to some subdivision of G_2.

Examples:

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Theorem (Kuratowski's theorem)

A graph is planar if and only if it does not contain a subgraph that is homeomorphic to K_5 or $K_{3,3}$.

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Example

The Petersen graph is not planar `https://upload.wikimedia.org/wikipedia/commons/0/0d/Kuratowski.gif`

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Week 9: Convex Polyhedra

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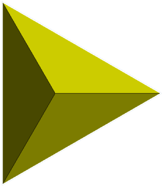
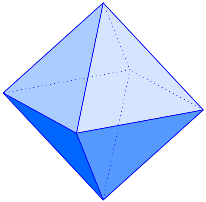
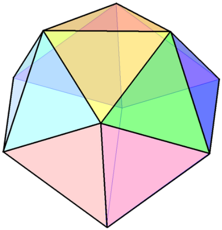
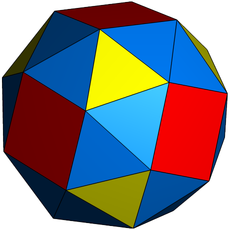
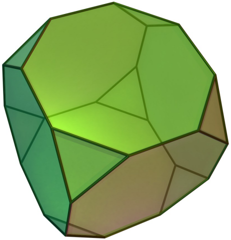
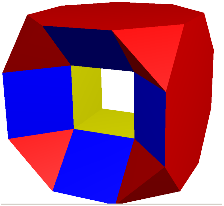
PART 2: Polyhedra

A *polyhedron* is a geometric solid made up of flat polygonal faces joined at edges and vertices.
A *convex polyhedron*, is one where any line segment connecting two points on the interior of the polyhedron must be entirely contained inside the polyhedron.

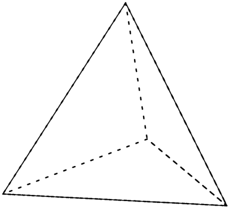
Examples:

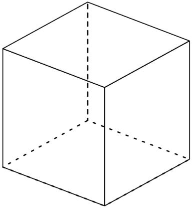
Source: WikiMedia Uniform polyhedron-43-s012.png, Truncatedhexahedron.jpg and Excavated_truncated_cube.png

A remarkable, and important fact, is that *every* convex polyhedron can be projected onto the plane without edges crossing.

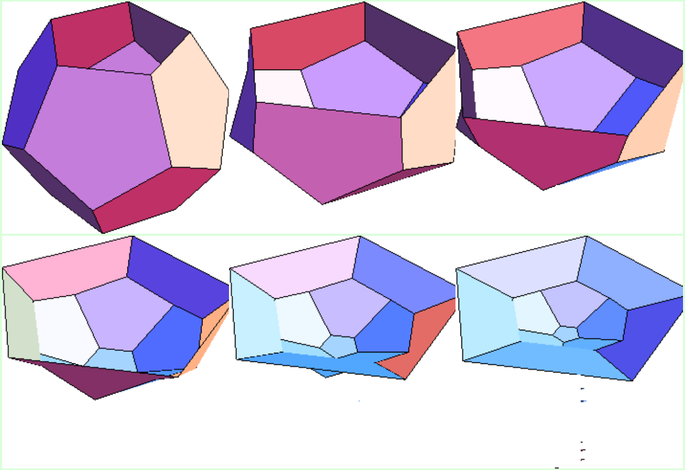
Example:



Example:

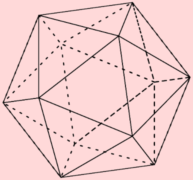


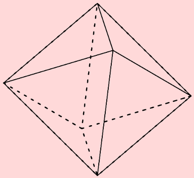
Example: the dodecahedron



Exercise

Give a planar projection of each of the following polyhedra.





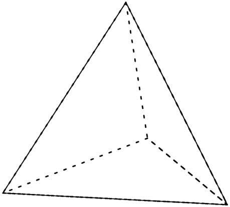
Now that we know every convex polyhedron can be represented as a planar graph, we can apply Euler's formula.

Euler's formula for polyhedra

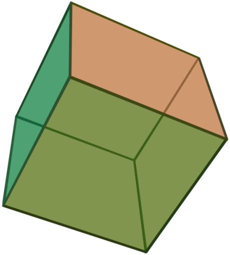
If a convex polyhedron has v vertices, e edges and f faces, then

$$v - e + f = 2$$

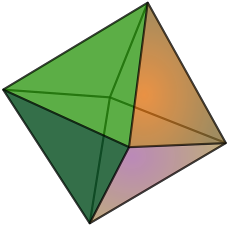
Example: the tetrahedron.



Example: the cube



Example: the octahedron



We now have two very powerful tools for studying convex polyhedra:

- **Euler's formula:** If a convex polyhedron has v vertices, e edges and f faces, then $v - e + f = 2$
- (The Handshaking Lemma) **The sum of the vertex degrees is $2|E|$:** let $G = (V, E)$ be a graph, with vertices $V = v_1, v_2, \dots, v_n$. Let $\deg(v_i)$ be the *"degree of v_i"*. Then

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|.$$

Example (See textbook, Section 4.2 (Polyhedra))

Show that there is no convex polyhedron with 11 vertices, all of degree 3?

See textbook, Example 4.2.3

Show that there is no convex polyhedron consisting of

- 3 triangles,
- 6 pentagons, and
- 5 heptagons (7-sided polygons).

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Week 9: Convex Polyhedra

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Week 9: Convex Polyhedra

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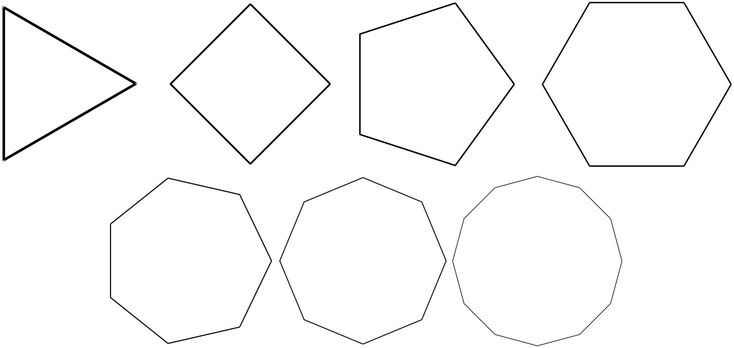
PART 3: Platonic Solids

Regular polyhedra - they are surprisingly few of them!

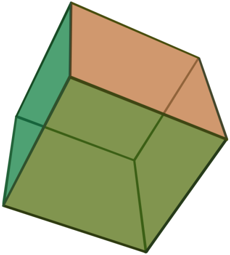
A **POLYGON** is a two-dimensional object. It is *regular* if all its sides are the same length:

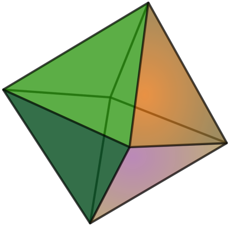


A *polyhedron* with the following properties is called **REGULAR** if

- All its faces are identical regular polygons.
- All its vertices have the same degree.

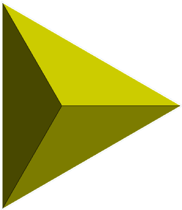
The convex regular polyhedra are also called *Platonic Solids*. Examples:

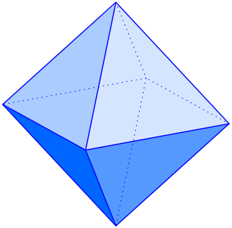


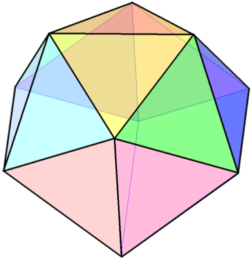


There are exactly 5 regular polyhedra

This fact can be proven using Euler's formula.
For full details, see the proof in the text book.
Here is the basic idea: we will only look at the case of polyhedra with triangular faces.







[All images here courtesy of Wikipedia]

MA284
Week 9: Convex Polyhedra

END OF PART 3

MA284
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Start of …

PART 4: Vertex Colouring

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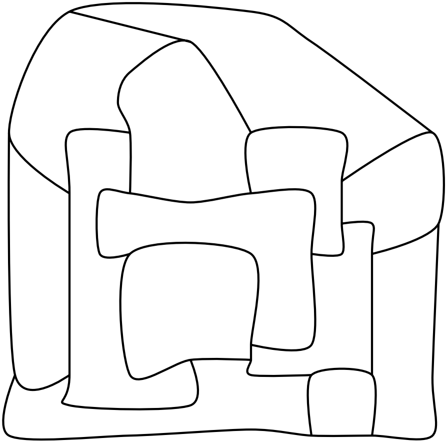

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- Three colours:

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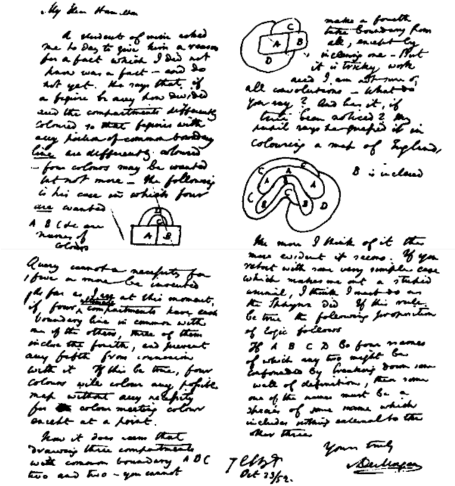

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To get a sense of *why* it might be true, try to draw a map that needs 5 colours.

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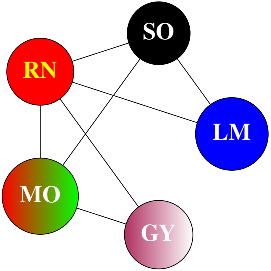
Our interest is not in trying to prove the Four Colour Theorem, but in how it is related to Graph Theory.

If we think of a map as a way of showing which regions share borders, then we can represent it as a *graph*, where

- A vertex in the graph corresponds to a region in the map;
- There is an edge between two vertices in the graph if the corresponds regions share a border.

Example:



Colouring regions of a map corresponds to colouring vertices of the graph. Since neighbouring regions in the map must have different colours, so too adjacent vertices in the graph must have different colours.

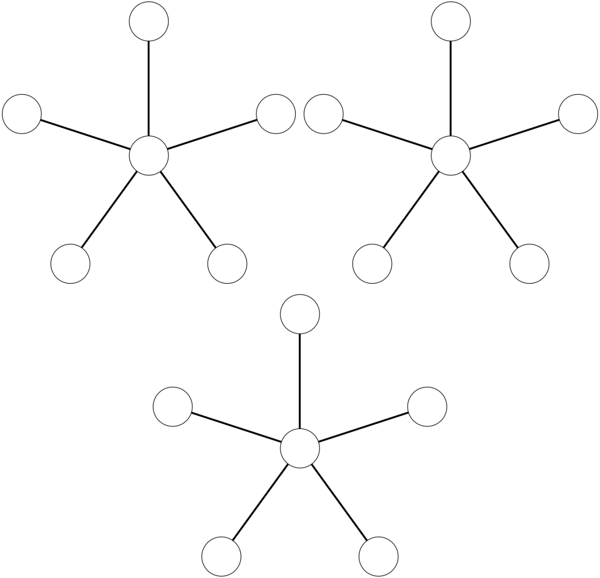
More precisely

Vertex Colouring: An assignment of colours to the vertices of a graph is called a *VERTEX COLOURING*.

Proper Colouring: If the vertex colouring has the property that adjacent vertices are coloured differently, then the colouring is called *PROPER*.

Lots different proper colourings are possible. If the graph as v vertices, the clearly at most v colours are needed. However, usually, we need far fewer.

Examples:



CHROMATIC NUMBER

The smallest number of colours needed to get a proper vertex colouring if a graph G is called the *CHROMATIC NUMBER* of the graph, written $\chi(G)$.

Example: Determine the Chromatic Number of the graphs C_2, C_3, C_4 and C_5.

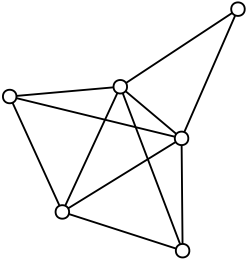
Example: Determine the Chromatic Number of the K_n and $K_{p,q}$ for any n, p, q.

In general, calculating $\chi(G)$ is not easy. There are some ideas that can help. For example, it is clearly true that, if a graph has v vertices, then

$$1 \leq \chi(G) \leq v.$$

If the graph happens to be *complete*, then $\chi(G) = v$. If it is *not* complete the we can look at *cliques* in the graph.

Clique: A *CLIQUE* is a subgraph of a graph all of whose vertices are connected to each other.



The **CLIQUE NUMBER** of a graph, G, is the number of vertices in the largest clique in G.

From the last example, we can deduce that

LOWER BOUND: The chromatic number of a graph is *at least* its clique number.

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We can also get a useful upper bound. Let $\Delta(G)$ denote the largest degree of any vertex in the graph, G,

UPPER BOUND: $\chi(G) \leq \Delta(G) + 1$.

Why? This is called **Brooks' Theorem**, and is Thm 4.5.5 in the text-book: `http://discrete.openmathbooks.org/dmoi3/sec_coloring.html`

MA284
Week 9: Convex Polyhedra

END OF PART 4

Most of these questions are taken from Levin's *Discrete Mathematics*.

Q1. Try to prove that $K_{3,3}$ is non-planar using *exactly* the same reasoning as that used to prove K_5 is non-planar. What does wrong? (The purpose of this exercise is to show that noting that $K_{3,3}$ has no 3-cycles is key. Also, we want to know that K_5 and $K_{3,3}$ are non-planar for different reasons).

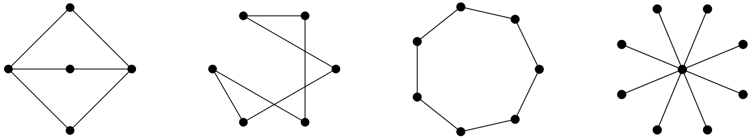
Q2. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

Q3. The graph G has 6 vertices with degrees 2,2,3,4,4,5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

Q4. Euler's formula ($v - e + f = 2$) holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has k components?

Q5. Prove that any planar graph with v vertices and e edges satisfies $e \leq 3v - 6$.

Q6. Which of the graphs below are bipartite? Justify your answers.

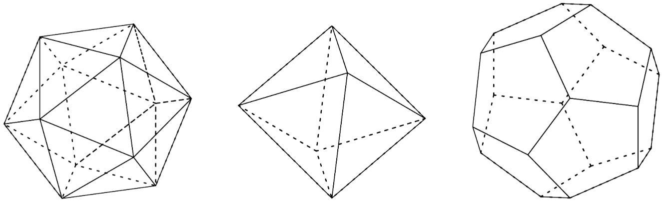


Q7. For which $n \geq 3$ is the graph C_n bipartite?

Q8. For each of the following, try to give two different unlabeled graphs with the given properties, or explain why doing so is impossible.

(a) Two different trees with the same number of vertices and the same number of edges. (A tree is a connected graph with no cycles).
(b) Two different graphs with 8 vertices all of degree 2.
(c) Two different graphs with 5 vertices all of degree 4.
(d) Two different graphs with 5 vertices all of degree 3.

Q9. Give a planar projection of each of the following polyhedra.



Q10. Show that there is only one regular convex polygon with square faces.

Q11. Show that there is only one regular convex polygon with pentagonal faces.

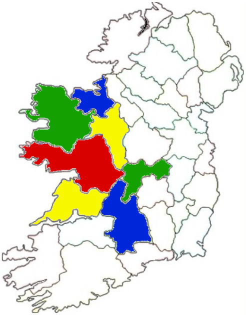
Q12. Could there be a regular polygon with faces that have more than 5 sides? Explain your answer.

MA284 : Discrete Mathematics

Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

Dr Kevin Jennings

9 and 11 November, 2021



See also Section 4.4 of Levin's *Discrete Mathematics.*

MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

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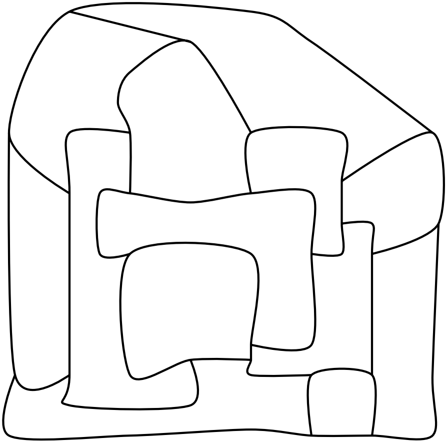

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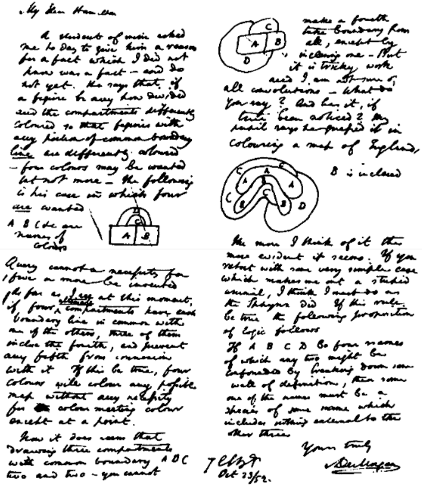

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MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

END OF PART 1

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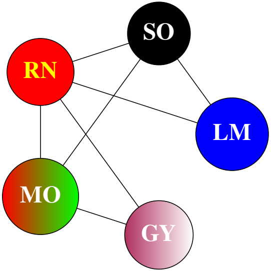

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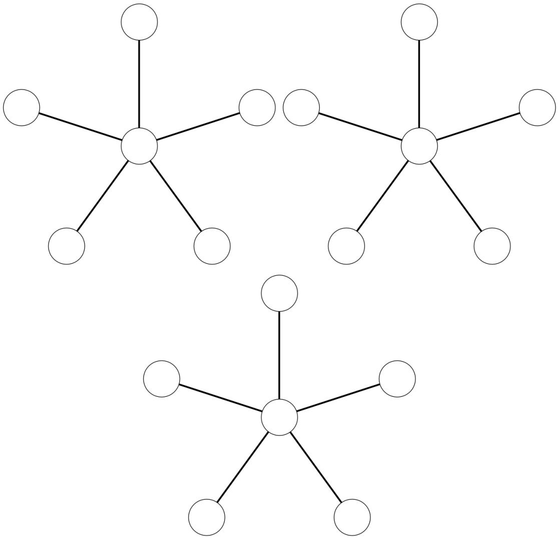

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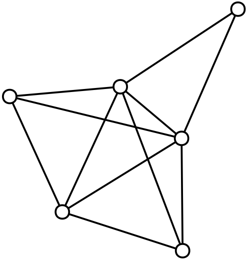

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MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

END OF PART 2

MA284
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Start of ...

PART 3: Algorithms for $\chi(G)$

In general, finding a proper colouring of a graph is *hard*.

There are some algorithms that are efficient, but not optimal. We'll look at two:

1. The *Greedy algorithm*.
2. The *Welsh-Powell algorithm.*

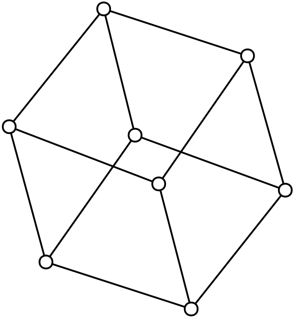
The **Greedy algorithm** is simple and efficient, but the result can depend on the ordering of the vertices.

Welsh-Powell is *slightly* more complicated, but can give better colourings.

The GREEDY ALGORITHM

1. Number all the vertices. Number your colours.
2. Give a colour to vertex 1.
3. Take the remaining vertices in order. Assign each one the lowest numbered colour, that is different from the colours of its neighbours.

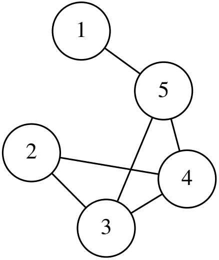
Example: Apply the GREEDY ALGORITHM to colouring the following graph (the cubical graph, Q_3):



Welsh-Powell Algorithm

1. List all vertices in decreasing order of their degree (so largest degree is first). If two or more have the same degree, list those any way.
2. Colour the first listed vertex (with first unused colour).
3. Work down the list, giving that colour to all vertices *not* connected to one previously coloured.
4. Cross coloured vertices off the list, and return to the start of the list.

Example: Colour this graph using both GREEDY and WELSH-POWELL:



Example

Seven one-hour exams, e_1, e_2, ... e_7, must be timetabled. There are students who must sit

(i) e_1 and e_5,

(ii) e_1 and e_7,

(iii) e_2, e_3, and e_6,

(iv) e_2, e_4, and e_7,

(v) e_3, e_5, and e_6,

(vi) e_4 and e_5

Model this situation as a vertex colouring problem, and find a scheduling that avoids timetable clashes and uses the minimum number of hours.

MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

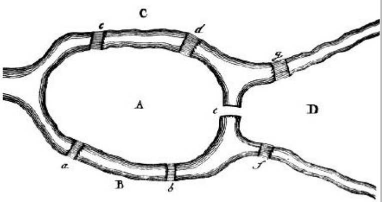
END OF PART 3

MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

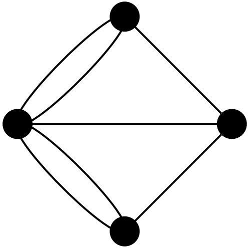
Start of ...

PART 4: Eulerian Paths and Circuits

We originally motivated the study of Graph Theory with the *Königsberg bridges* problem: find a route through the city that crosses every bridge once and only once:



The seven bridges of Königsberg



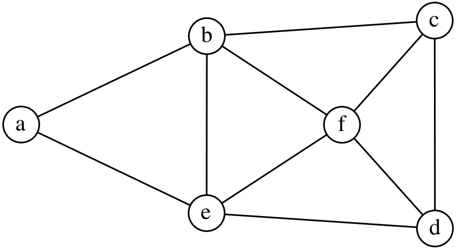
We'll now return to this problem, and show that there is no solution. First we have to re-phrase this problem in the setting of graph theory.

Recall (from Week 8) that a **PATH** in a graph is a sequence of adjacent vertices in a graph.

Eulerian Path

An **EULERIAN PATH** (also called an *Euler Path* and an *Eulerian trail*) in a graph is a path which uses every edge exactly once.

Example:

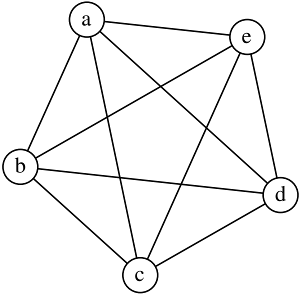


Recall from Week 8 that a **circuit** is a path that begins and ends at that same vertex, and no **edge** is repeated...

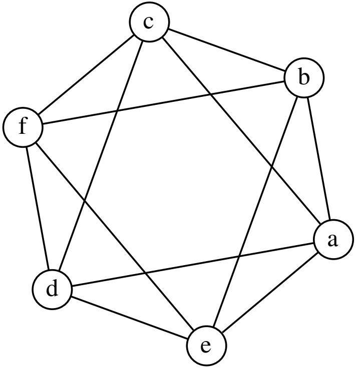
Eulerian Circuit

An **EULERIAN CIRCUIT** (also called an *Eulerian cycle*) in a graph is an *Eulerian* path that starts and finishes at the same vertex.
If a graph has such a circuit, we say it is *Eulerian*.

Example 1 (K_5):



Example 2: Find an Eulerian circuit in this graph:

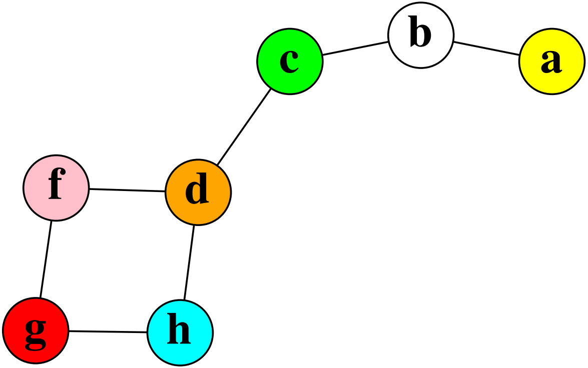


Of course, not every graph as an Eulerian circuit, or, indeed, and Eulerian path.

Here are some extreme examples:

It is possible to come up with a condition that *guarantee* that a graph has an *Eulerian path*, and, addition, one that ensures that it has an *Eulerian circuit*.

To begin with, we'll reason that the following graph could *not* have an Eulerian circuit, although it *does* have an Eulerian path:



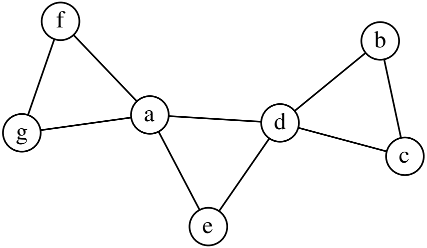
Suppose, first, the we have a graph that **does have an Eulerian circuit**. Then for every edge in the circuit that "*exits*" a vertex, there is another that "*enters*" that vertex. So every vertex must have even degree.

Example (W3)

In fact, a stronger statement is possible.

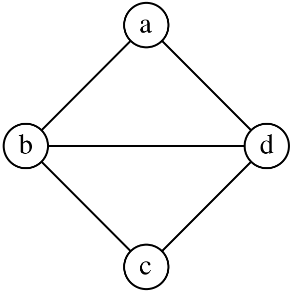
A graph has an **EULERIAN CIRCUIT** if and only if every vertex has even degree.

Example: Show that the following graph has an *Eulerian circuit*



Next suppose that a graph **does not have an Eulerian circuit**, but does have an **Eulerian Path**. Then the degree of the "start" and "end" vertices must be odd, and every other vertex has even degree.

Example:

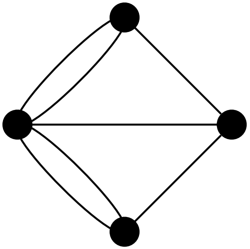


To summarise:

Eulerian Paths and Circuits

- A graph has an **EULERIAN CIRCUIT** *if and only if* the degree of every vertex is even.
- A graph has an **EULERIAN PATH** *if and only if* it has either **zero** or **two** vertices with odd degree.

Example: The *Königsberg bridge* graph does not have an Eulerian path:



Example (MA284, 2020/21 Semester 1 Exam)

Let $G = (V, E)$, where $V = \{a, b, c, d, e, f, g\}$, and
$E = \{\{a, b\}, \{a, g\}, \{b, c\}, \{b, d\}, \{b, g\}, \{c, d\}, \{d, e\}, \{e, f\}, \{e, g\}, \{f, g\}$.
Does G admit an Eulerian Path and/or Circuit? If it does, exhibit one. If not, explain why.

MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

END OF PART 4

MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

Start of ...

PART 5: Hamiltonian Paths and Cycles

Next we'll look at a closely related problem: finding paths through a graph that visit every vertex exactly once.

These are called **HAMILTONIAN PATH**, and are named after the (very famous) William Rowan Hamilton, the Irish mathematician, who invented a board-game based on the idea.

Hamilton's Icosian Game (Library of the Royal Irish Academy)

Try playing online: `https://www.geogebra.org/m/u3xggkcj`

Definition (HAMILTONIAN PATH)

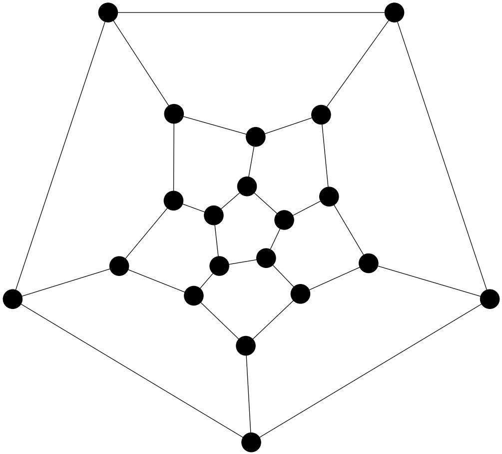
A path in a graph that visits every vertex exactly once is called a **HAMILTONIAN PATH**.

Hamiltonian Cycles

Recall that a **CYCLE** is a path that starts and finishes at the same vertex, but no other vertex is repeated.
A **HAMILTONIAN CYCLE** is a cycle which visits the start/end vertex twice, and every other vertex exactly once.
A graph that has a Hamiltonian cycle is called a **HAMILTONIAN GRAPH**.

Examples:

This is the graph based on Hamilton's Icosian game. We'll find a Hamilton path. Can you find a Hamilton cycle?



Important examples of Hamiltonian Graphs include:

- *cycle graphs*;
- *complete graphs*;
- *graphs of the platonic solids*.

In general, the problem of finding a Hamiltonian path or cycle in a large graph is **hard** (it is known to be NP-complete). However, there are two relatively simple *sufficient conditions* to testing if a graph is Hamiltonian.

1. Ore's Theorem

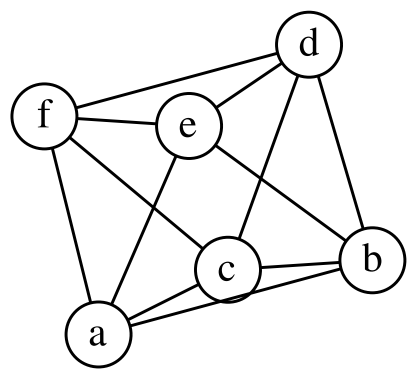
A graph with v vertices, where $v \geq 3$, is *Hamiltonian* if, for every pair of non-adjacent vertices, the sum of their degrees $\geq v$.

2. Dirac's Theorem

A (simple) graph with v vertices, where $v \geq 3$, is *Hamiltonian* if every vertex has degree $\geq v/2$.

Example

Determine whether or not the graph illustrated below is Hamiltonian, and if so, give a Hamiltonian cycle:



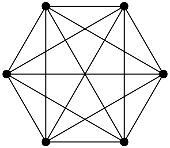
MA284
Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

END OF PART 5

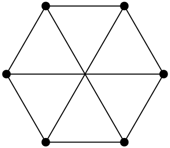
Q1. (Textbook) What is the smallest number of colors you need to properly color the vertices of $K_{4,5}$? That is, find the chromatic number of the graph.

Q2. Determine the chromatic number of each of the following graphs, and give a colouring for that achieves it.

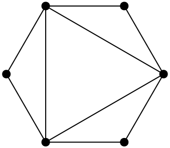
(i)



(ii)



(iii)



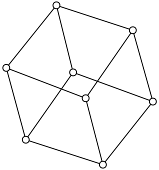
Q3. For each of the following graphs, determine if it has an Eulerian path and/or circuit. If not, explain why; otherwise give an example.

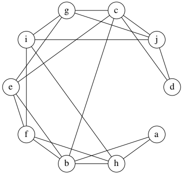
(a) K_n, with n even.

(b) $G_1 = (V_1, E_1)$ with $V_1 = \{a, b, c, d, e, f\}$,
$E_1 = \{\{a, b\}, \{a, f\}, \{c, b\}, \{e, b\}, \{c, e\}, \{d, c\}, \{d, e\}, \{b, f\}\}$.

(c) $G_2 = (V_2, E_2)$ with $V_2 = \{a, b, c, d, e, f\}$,
$E_2 = \{\{a, b\}, \{a, f\}, \{c, b\}, \{e, b\}, \{c, e\}, \{d, c\}, \{d, e\}, \{b, f\}, \{b, d\}\}$.

Q4. For each of the following graphs, determine if it has an *Eulerian path* and/or *Eulerian circuit*. If so, give an example; if not, explain why.

$G_1 =$ 

$G_2 =$ 

Q5. *Given a graph $G = (V, E)$, its* ***compliment*** *is the graph that has the same vertex set, V, but which has an edge between a pair of vertices* ***if and only if*** *there is no edge between those vertices in G.*

Sketch of of the following graphs, and their complements:

(i) K_4, (ii) C_4, (iii) P_4, (iv) P_5.

Q6. Which of the following graphs are isomorphic to their own complement ("self-complementary")?

(i) K_4, (ii) C_4, (iii) P_4, (iv) P_5.

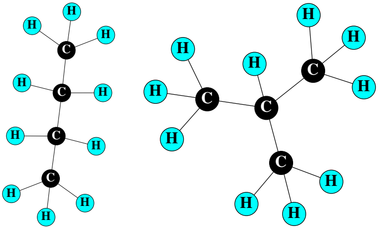
Q7. Show that $K_{3,3}$ has Hamiltonian, but $K_{2,3}$ is not.

MA284 : Discrete Mathematics

Week 11: Trees

16 and 18 November, 2022

Dr Kevin Jennings (slides thanks to Dr Niall Madden), University of Galway



See also Section 4.2 of Levin's *Discrete Mathematics.*

MA284
Week 11: Trees

Start of ...

PART 1: Trees

There's an important class of graphs that do not contain circuits: **TREES**. The mathematical study of trees dates to at least 1857, when Arthur Cayley used them to study certain chemical compounds.

They are used in many mathematical models of decision making (such as Chess programmes), and in designing algorithms for data encoding and transmission.

Definition: ACYCLIC/FOREST

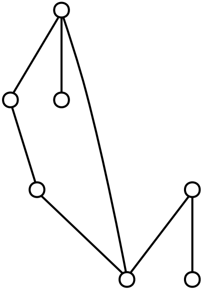
A graph that has no circuits is called **ACYCLIC** or a "forest".

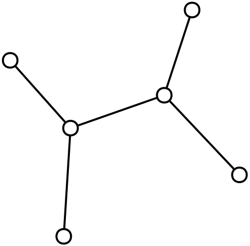
Definition: TREE

A **TREE** is a connected, acyclic graph.

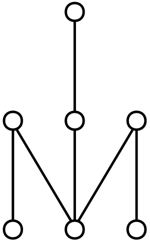
Examples:

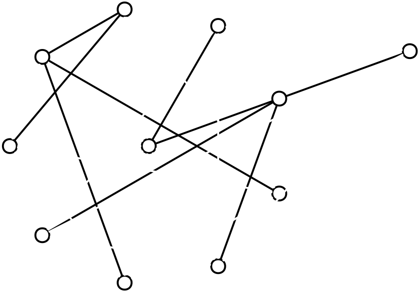
Which of the following are graphs of trees?





Which of the following are graphs of trees?





Another characterisation of trees

A graph is a tree if and only if there is a **unique path** between any two vertices.

If T is a tree, then $e = v - 1$

If T is a tree (i.e., a connected acyclic graph) with v vertices, then it has $v - 1$ edges. (We will see that the converse of this statement is also true).

(See also Prop 4.2.4 in the textbook).

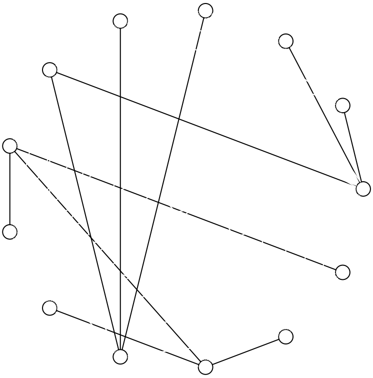
It can be difficult to determine if a very large graph is a tree just by inspection. If we know it has no cycles, then we need to verify that it is connected. The following result (the converse of the previous one) can be useful.

If $e = v - 1$, then T is a tree

If graph with v vertices has *no* cycles, and has $e = v - 1$ edges, then it is a tree.

Example

The following graph has no cycles. Determine how many components it has. Is it a tree?



MA284
Week 11: Trees

END OF PART 1

MA284
Week 11: Trees

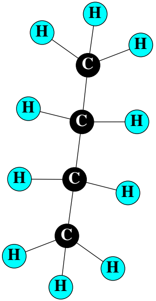
Start of ...

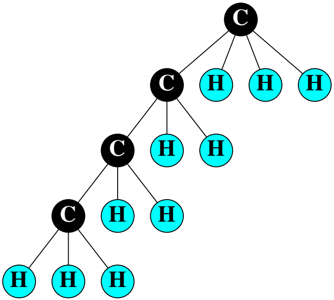
PART 2: Applications

There are many, *many* applications, of trees in mathematics, computer science, and the applied sciences. As already mentioned, the mathematical study of trees began in Chemistry.

Example: *Saturated hydrocarbon* isomers

Saturated hydrocarbon isomers(alkane) are of the form C_nH_{2n+2}. They have n carbon atoms, and $2n + 2$ hydrogen atoms. The carbon atoms can bond with 4 other atoms, and the hydrogens with just one. Show that the graph of all such isomers are trees.


H
H H
C
H C H
H C H
C
H H
H


C
C H H H
C H H
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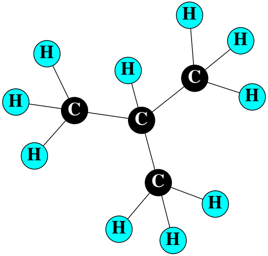
H H H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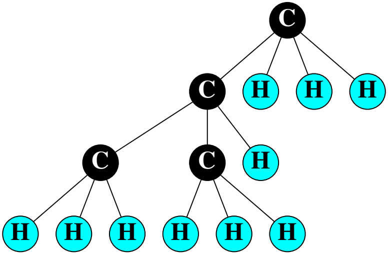

C
C H H H
C C H
H H H H H H

A **DECISION TREE** is a graph where each node represents a possibility, and each branch/edge from that node is a possible outcome.

Example

Pancho and Lefty played a chess match in which there were no drawn games. The first player to win three games in a row or a total of four games won the match.
Pancho won the first game and the person who won the second game also won the third game.
Construct an appropriate tree diagram to find the number of ways in which the match may have proceeded.

(PTO)

Puzzle

You have **eight** identical-looking coins, but one is a counterfeit and lighter than the rest. You have a balance scale. Show that you can find the counterfeit one with just **two** weighing.
How many weighings are needed for **nine** coins? And **ten**?

MA284
Week 11: Trees

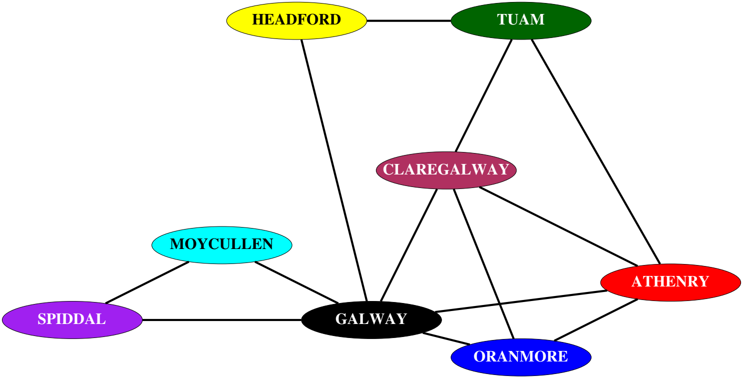
END OF PART 2

MA284
Week 11: Trees

Start of ...

PART 3: Spanning Trees

Consider the road system shown below

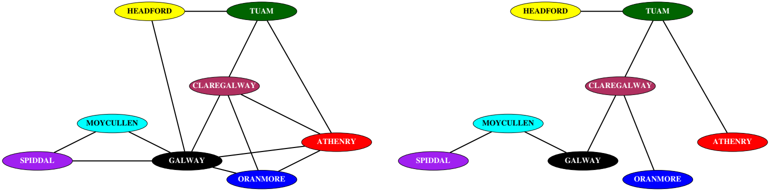


Suppose there has been severe flooding, and Galway County Council can only keep a small number of roads open? Which ones should they choose, so that one can travel between any pair of towns?

Definition: SPANNING TREE

Given a (simple) graph G, a **SPANNING TREE** of G is a subgraph of G that

- is a tree, and
- contains every vertex of G.



Lots of other spanning trees are possible, and there are numerous ways of finding them...

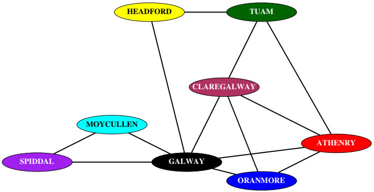
Lots of other spanning trees are possible, and there are numerous ways of finding them. Here are two:

Algorithm 1

(i) Identify a cycle in the graph

(ii) Delete an edge in that cycle, taking care not to disconnect the graph.

(iii) Keep going until all cycles have been removed.

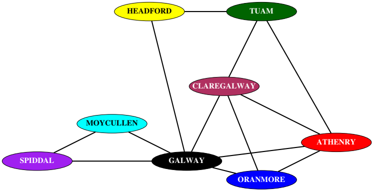


Algorithm 2

(i) Start with just the vertices of the graph (no edges).

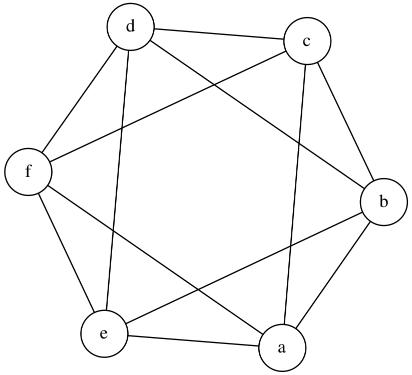
(ii) Add an edge from the original graph, as long as it does not form a cycle.

(iii) Stop when the graph is connected.



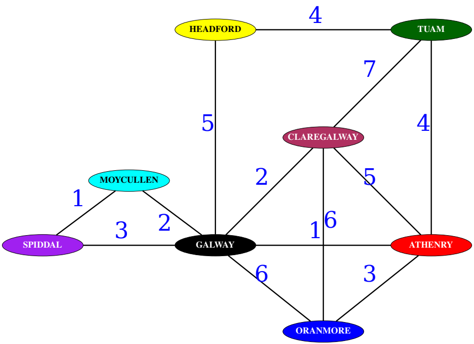
Example

Find a spanning tree in the connected graph illustrated below:



For many applications, we need to consider a wider class of graphs: **weighted graphs**. We won't go into details, but we'll see this with a slight modification of our initial example.

Suppose that one can estimate how long it would take to fix a section of road, as shown by the **weights** on the edges.
Which ones should we fix, so that one can travel between any pair of towns **as soon as possible**?



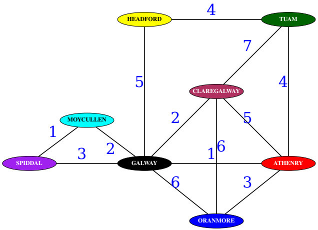
A **minimum spanning tree** is a spanning tree with the minimum possible total edge weight. Minimum spanning trees exist and there are various algorithms to find them.

Algorithm: Kruskal's

(i) Start with just the vertices;

(ii) Add the edge with the least weight that does not form a cycle.

(iii) Keep going until the graph is connected.

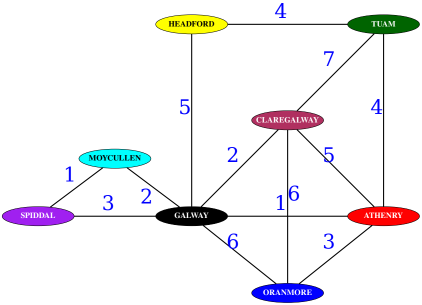


Algorithm: Prim's algorithm

(i) Choose a(ny) vertex from the original graph.

(ii) Add the edge incident to that vertex that has least weight and does not create a cycle.

(iii) Stop when you reached all the vertices of the original graph.



There's a lot of maths involved in planning public transport, roads and all that.

Here's an article on *The Maths of public transport in Galway*, by my colleague Michael Mc Gettrick:

`https://www.rte.ie/brainstorm/2020/0204/1113099-the-maths-of-public-transport/`

There are many other applications of trees that, regrettably, we do not have time to cover. The most important of these include

- minimum spanning trees.
- the study of search algorithms modelled as trees;
- decision tress (like the puzzle from Slide 17);
- compiler syntax;
- Financial modelling: e.g., binomial methods for option pricing;
- The "Good Will Hunting" Problem (draw all homomorphically irreducible trees with $v = 10$ vertices).

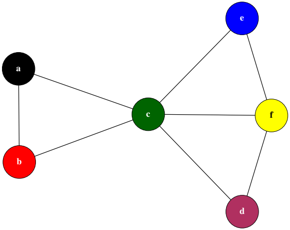
Q1. (See Exer 1 in §4.2 of text). Which of the following graphs are trees?

(a) $G = (V, E)$ with with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.
(b) $G = (V, E)$, with with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.
(c) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}$.
(d) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{d, e\}\}$.

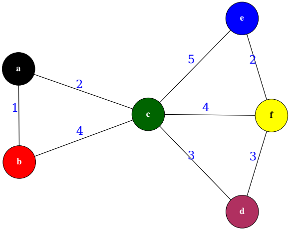
Q2. (See Q2 in Section 4.2 of text-book). For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

(a) $(4, 1, 1, 1, 1)$
(b) $(3, 3, 2, 1, 1)$
(c) $(2, 2, 2, 1, 1)$
(d) $(4, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$.

Q3. Give at least 3 different spanning trees of the graph shown below.



Q4. Give a minimum spanning tree of the weighted graph shown below.



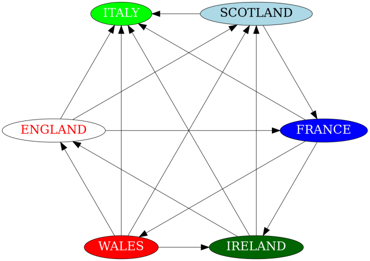
MA284 : Discrete Mathematics

Week 12: Matrices and Review

23 and 25 November, 2022

Dr Kevin Jennings (slides by Dr Niall Madden, mistakes are Kevin's),

University of Galway



- **Assignment 4** is still open. Deadline is *5pm, Thursday 24 November* (note slight extension).
- **Assignment 5** is due: Deadline is **5pm Tuesday, 25 November 2022**.

MA284
Week 12: Matrices and Review

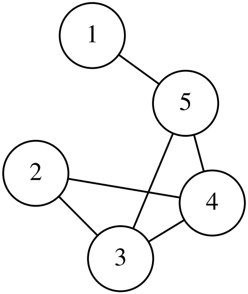
Start of ...

PART 1: Matrices

In this short section we study other ways to represent matrices mathematically. These have important applications in computing with graphs and networks.

In a practical setting, a graph must be stored in some computer-readable format. One of the most common is an **adjacency matrix**. If the graph has n vertices, labelled $\{1, 2, \ldots, n\}$, then the adjacency matrix is an $n \times n$ **binary** matrix, A, with entries

$$a_{i,j} = \begin{cases} 1 & \text{vertex } i \text{ is adjacent to } j \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Properties of the adjacency matrix

- The adjacency matrix of a graph is **symmetric**.
- If $B = A^k$, then $b_{i,j}$ is the number of paths of length k from vertex i to vertex j.
- We can work out if a graph is connected by looking at the eigenvalues of A.
- If the graphs G and H are isomorphic, and have adjacency matrices A_G and A_H, respectively, then there is a permutation matrix, P, such that $PA_GP^{-1} = A_H$.

Unfortunately, we don't have time to prove these properties, but reviewing some examples can still be very instructive.

Example

Sketch a graph represented by the following adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. What is the order of the graph?
2. How many edges does the graph has?
3. Is the graph a pseudograph (i.e., a graph with loops), a multigraph, or a simple graph?
4. What is the degree of Vertex 2?
5. What is the degree of Vertex 4?

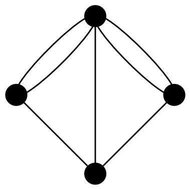
In the previous example, you were asked to determine if the graph was simple, or a multigraph, or had loops.

The adjacency matrix idea is easily extended to allow for the last two cases:

- For a multigraph, a_{ij} is the number of edges joining vertices i and j.
- For a pseudograph (graph with loops), $a_{ii} = 1$ means there is an edge from a vertex to itself. vertices i and j.

Example

Give the adjacency matrix of the Konigsberg Bridges graph:

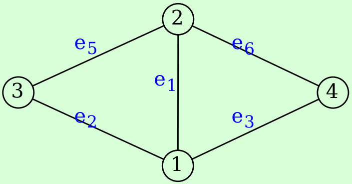


Graphs can also be represented by an **Incidence matrix**

- If the graph has v vertices, and e edges, then it is an $v \times e$ binary matrix.
- The rows represent vertices
- The columns represent edges.
- If the matrix is $B = (b_{i,j})$ then $b_{ik} = 1$ means that vertex i is incident to edge j.

Example

Give the incidence matrix of the following graph:



MA284
Week 12: Matrices and Review

END OF PART 1

MA284
Week 12: Matrices and Review

Start of ...

PART 2: Distance Matrices

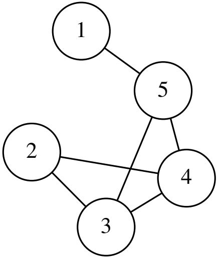
In this section we quantify how "far" two vertices are from each other in a graph

Recall that a **path** is a sequence of edges from one vertex to another. Usually, there are multiple paths between any pair of vertices.

Definition: DISTANCE

The **distance** between two vertices, u and v, in a connected graph is the length of a *shortest path* between u and v, and is written $d(u, v)$. (Warning: notation is easily confused with the degree of a vertex.) (This is also called **geodesic distance** or **shortest-path distance**).

Usually we represent all the distances between vertices in a graph as a matrix:



	1	**2**	**3**	**4**	**5**
1	0	3	2	2	1
2	3	0	1	1	2
3	2	1	0	1	1
4	2	1	1	0	1
5	1	2	1	1	0

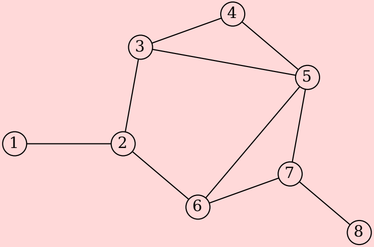
One we have the idea of **distance** we can then define

- **Eccentricity of a vertex:** the greatest distance between that vertex and any other in the graph.
- **Radius of a graph:** the minimum eccentricity of any vertex.
- **Diameter of a graph:** the maximum eccentricity of any vertex. So this is also the maximum entry in the distance matrix.

Example

Consider the following graph.

- Write down the distance matrix for this graph,
- Use the distance matrix to determine the eccentricity of each vertex.
- Determine the radius and diameter of the graph.



MA284
Week 12: Matrices and Review

END OF PART 2

MA284
Week 12: Matrices and Review

Start of ...

PART 3: Additional Topics

Other topics in combinatorics and graph theory that we have not yet covered. The most interesting (to my mind are):

[ie Dr Niall Madden's preference - I might speak about music tomorrow but I'll leave these notes here in case students are interested in the software Niall used to prepare these slides.]

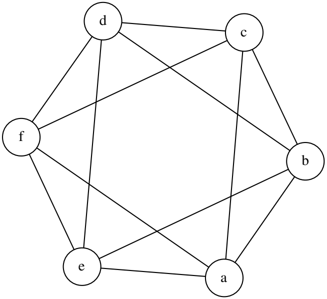
1. Directed Graphs
2. Visualisation of graphs;
3. Algorithms, like determining if a graph is connected, or finding the shortest path between two vertices...
4. The Graph Laplacian;
5. Centrality analysis.
6. and many, many, more...

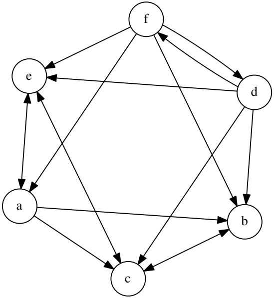
We'll finish now with a short presentation on the first 4 of these.

Graphs often represent networks, such as the road network we had earlier, or social networks. So far, we have had that, if vertex a is adjacent to vertex b, then b is adjacent to a.

In many situations, this is not reasonable:

- a city road system might have a one-way system;
- on a social network, you might follow someone who does not follow you back.



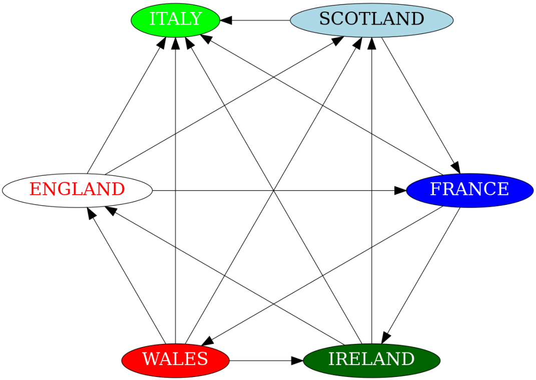


Definition: Directed Graph (=Digraph)

A **directed graph** (also called a "digraph") is one where the edge set is composed of **ordered** pairs, called "**directed edges**" ("arcs"). If $(a, b) \in E$, we say "there is a directed edge from vertex a to vertex b".
When sketching the graph, we indicate the direction of the edge by an arrow.

1. The number of directed edges starting at vertex a is the **out degree** of a.
2. The number of directed edges ending at vertex a is the **in degree** of a.
3. The adjacency matrix is not usually symmetric.

Example: Graph of a tournament (2021 Senior Men's 6 Nations Rugby). A directed edge from Team A to Team B means Team A beat Team B. In this case, the graph is used only to summarise the outcome. However, various algorithms for ranking teams use methods based on graphs.



Because graphs are key to understanding and analysing networks of any type, there are numerous computational tools for working with graphs.

Ones that I have used in the course of this module – mainly for generating images are

- Graphviz, which combines a language called "dot" with a set of tools for converting these graphs to images.
- SageMath, a free computer algebra system, and
- NetworkX, a Python-based system for analysing graphs and networks.

Graphviz

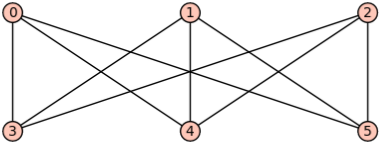
Making K_5

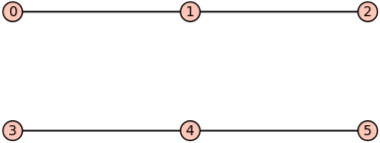
```
graph  {
1 -- {2 3 4 5};
2 -- {3 4 5};
3 -- {4 5};
4 -- {5};
}
```

To generate a image from this you can either install GraphViz on your own computer, or use an online tool such as `http://www.webgraphviz.com/` or `https://dreampuf.github.io/GraphvizOnline`

SageMath

```
G = graphs.CompleteBipartiteGraph(3, 3)
G.show()
H = G.complement()
H.show()
```

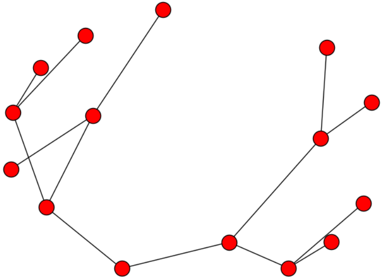




You can do a lot more than this with Sage, including applying numerous algorithms for, say, computing the Chromatic Number of a graph, or finding a minimum spanning tree

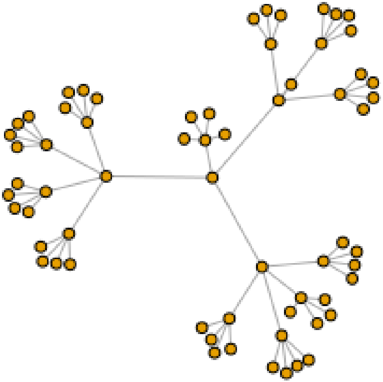
NetworkX

```
import matplotlib.pyplot as plt
import networkx as nx
G = nx.balanced_tree(2,3)
nx.draw(G), plt.show()
```



R with the igraph library

```
library(igraph)
tree <- make_tree(64, 4, mode="undirected")
plot(tree, vertex.size=6, vertex.label=NA)
```



MA284
Week 12: Matrices and Review

END OF PART 3

MA284
Week 12: Matrices and Review

Start of ...

PART 4: Review

Th set of topics that we studied includes:

1. The additive and multiplicative principles;
2. Sets; the Principle of **Inclusion/Exclusion** (PIE) and its applications;
3. **Binomial Coefficients** (& lattice paths, bit-strings, & Pascal's triangle);
4. Permutations and Combinations;
5. **Stars and Bars**, and the NNI Equations and Inequalities;
6. Algebraic V **Combinatorial** Proofs;
7. Derangements;
8. Counting functions;

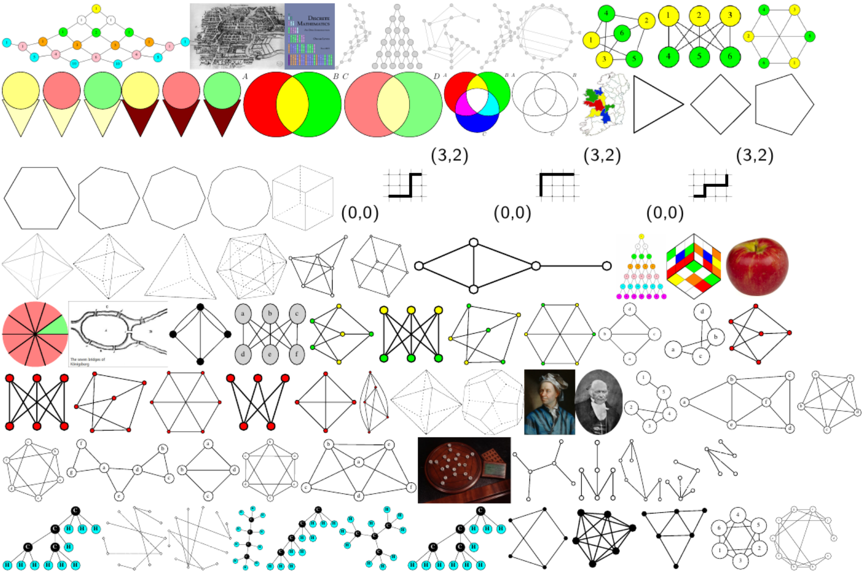
9. Graph Theory: motivation and basic definitions;
10. Isomorphisms between graphs.
11. Important families of graphs (Cycle graphs, K_n, $K_{n,n}$, etc.)
12. Planar & non-planar graphs; chromatic numbers, **Euler's formula**,
13. Convex polyhedra, and Platonic solids;
14. Graph Colouring; Greedy and Welsh-Powell algorithms;
15. Eulerian and Hamiltonian graphs;
16. Trees, including spanning trees, and decision trees.
17. Matrices of Graphs.

There are **8** questions on the final MA284 exam: you should attempt *all* eight. 4 questions are worth 13 marks, and 4 are worth 12.

. . . .

Tips:

- The questions on the exam are roughly in the order in which we covered the topics in class.
- 4 questions are on combinatorics, and 4 are on graph theory.
- The Principles of Addition, Multiplication, and Inclusion/Exclusion are essential to most of the combinatorics questions.
- Good idea to review the homework exercises.
- For graph theory, you need to know how to
  - sketch a graph given the edge and vertex sets;
  - determine if the graph is, e.g., bipartite, planar, connected, ...
  - find an Eulerian path/circuit.
  - compute the chromatic number
  - calculate the radius and diameter.


Discrete Mathematics
A
B
C
D
(3,2)
(0,0)
(3,2)
(0,0)
(3,2)
(0,0)

I'm especially grateful to Drs Niall Madden in particular and Angela Carnevale who pitched a very interesting cross-section of discrete maths as a syllabus and who prepared such wonderful slides.

Good luck with your exams and I hope you've enjoyed studying this introduction to discrete maths. We have plenty more graph theory in some of our final year modules eg Networks and in some Applied Maths modules eg Modelling.

Q1. Write down the adjacency for each of the following graphs.

(a) $G = (V, E)$ with with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.
(b) K_5
(c) C_5
(d) $K_{3,3}$

Q2. Determine if the following matrices represent adjacency matrices of simple connected graphs. If not, explain why.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Q3. Write down the distance matrix for the followingt graph, and use it to determine the eccentricity of each vertex. Determine the radius and diameter of the graph.