CS4423: Problem Set 1

These exercises are to help you master material covered in classes. You don't have to submit your work. However, Assignment 1, which will be posted later in Week 4, will include questions based on some of these exercises.

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Q1. For what values of n is K_n bipartite?

Q2. For what values of m and n is $K_{m,n}$ bipartite?

Q3. For what values of n is P_n bipartite?

Q4. For what values of n is C_n, the cycle graph on $n \geqslant 3$ nodes, bipartite?

Q5. Let G be the graph on the set of nodes $\{1, 2, 3, 4, 5, 6\}$ with edges $1-2$, $1-3$, 2–4, $3-4$, $3-6$, $4-5$, $4-6$. Draw the graph G. Is G bipartite? Justify your answer. (Note: writing "$a-b$ is an edge in G" is the same as saying (a, b) is an element of its edge set).

Q6. At a party with $n = 5$ people, some people know each other already while others don't. Each of the 5 guests is asked how many friends they have at this party. Two report that they have one friend each. Two other guests have two friends each, and the fifth guest has three friends at the party. Understanding friendship as a symmetric relation, is this network possible? Why, or why not? (*Hint: recall that the sum of all node degrees is twice the number of edges in the graph*).

Q7. We say two graphs are *equal* if they have the same node and edge sets. We say they are *isomorphic* if there is a relabling of their nodes that makes them equal. Verify that C_5 is isomorphic to its complement.

Q8. Convince yourself that C_n is always isomorphic to $L(C_n)$, the line graph of C_n.

Q9. Let G be any graph of order n. Let $\bar{G}$ be its compliment. Call their adjacency matrices A_G and $A_{\bar{G}}$, respectively. Let H be the graph with adjacency matrix $A_G + A_{\bar{G}}$. By what name is H more commonly known?

Q10. Let P_n be the path graph on $n \geqslant 2$ vertices. There is exactly one n for which P_n is isomorphic to its complement, $\bar{P_n}$. What value of n is that? Show that there are no other values of n for which P_n is isomorphic to $\bar{P_n}$

Q11. Is the Petersen graph bipartite? Explain your answer.

Q12. Write down the adjacency matrix, A of $K_{2,3}$. Compute A^2 and A^3. Use A^3 this to verify that $K_{2,3}$ has no triangles (3-cycles).

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Theory for the Q13 and Q14 will be covered in lectures in Week 4.

Q13. Consider the graph, G, shown in Figure 1.

(a) Write down the node set, V, edge set E, and adjacency matrix A for this graph.

(b) Find a permutation matrix, P, such that PAP^T is structured like:

$$PAP^T = \begin{pmatrix} A_{11} & O_{12} \\ O_{12}^T & A_{22}. \end{pmatrix}$$

where O_{12} is a 5×3 matrix of zeros.

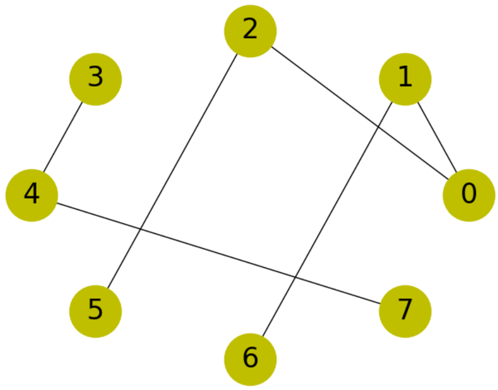

Figure 1: Graph for Q13

(c) Show that $(PAP^T)^k = \begin{pmatrix} A_{11}^k & O_{12} \\ O_{12}^T & A_{22}^k. \end{pmatrix}$ for all k, and conclude that G is not connected.

Q14. Consider the bipartite graph, shown in Figure 1. Construct the projection of it onto the sets $\{a, b, c, d, e\}$. Sketch the resulting graph.

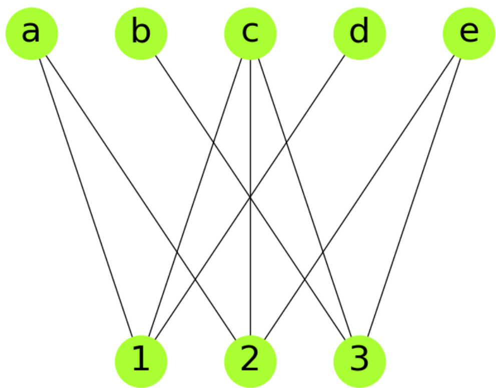

Figure 2: Graph for Q14