MA284 : Discrete Mathematics

Week 12: Matrices and Review

23 and 25 November, 2022

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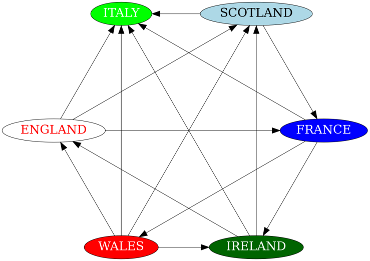
University of Galway



- **Assignment 4** is still open. Deadline is *5pm, Thursday 24 November* (note slight extension).

- **Assignment 5** is due: Deadline is **5pm Tuesday, 25 November 2022**.

MA284
Week 12: Matrices and Review

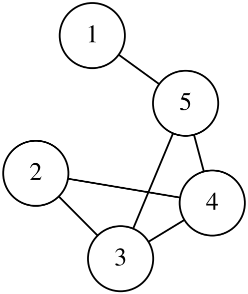
Start of ...

PART 1: Matrices

In this short section we study other ways to represent matrices mathematically. These have important applications in computing with graphs and networks.

In a practical setting, a graph must be stored in some computer-readable format. One of the most common is an **adjacency matrix**. If the graph has n vertices, labelled $\{1, 2, \ldots, n\}$, then the adjacency matrix is an $n \times n$ **binary** matrix, A, with entries

$$a_{i,j} = \begin{cases} 1 & \text{vertex } i \text{ is adjacent to } j \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Properties of the adjacency matrix

- The adjacency matrix of a graph is **symmetric**.
- If $B = A^k$, then $b_{i,j}$ is the number of paths of length k from vertex i to vertex j.
- We can work out if a graph is connected by looking at the eigenvalues of A.
- If the graphs G and H are isomorphic, and have adjacency matrices A_G and A_H, respectively, then there is a permutation matrix, P, such that $PA_GP^{-1} = A_H$.

Unfortunately, we don't have time to prove these properties, but reviewing some examples can still be very instructive.

Example

Sketch a graph represented by the following adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. What is the order of the graph?
2. How many edges does the graph has?
3. Is the graph a pseudograph (i.e., a graph with loops), a multigraph, or a simple graph?
4. What is the degree of Vertex 2?
5. What is the degree of Vertex 4?

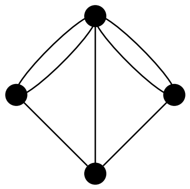
In the previous example, you were asked to determine if the graph was simple, or a multigraph, or had loops.

The adjacency matrix idea is easily extended to allow for the last two cases:

- For a multigraph, a_{ij} is the number of edges joining vertices i and j.
- For a pseudograph (graph with loops), $a_{ii} = 1$ means there is an edge from a vertex to itself. vertices i and j.

Example

Give the adjacency matrix of the Konigsberg Bridges graph:

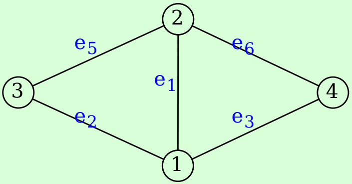


Graphs can also be represented by an **Incidence matrix**

- If the graph has v vertices, and e edges, then it is an $v \times e$ binary matrix.
- The rows represent vertices
- The columns represent edges.
- If the matrix is $B = (b_{i,j})$ then $b_{ik} = 1$ means that vertex i is incident to edge j.

Example

Give the incidence matrix of the following graph:



MA284
Week 12: Matrices and Review

END OF PART 1

MA284
Week 12: Matrices and Review

Start of ...

PART 2: Distance Matrices

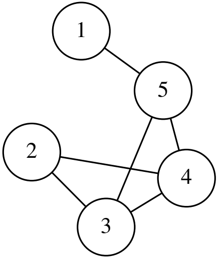
In this section we quantify how "far" two vertices are from each other in a graph

Recall that a **path** is a sequence of edges from one vertex to another. Usually, there are multiple paths between any pair of vertices.

Definition: DISTANCE

The **distance** between two vertices, u and v, in a connected graph is the length of a *shortest path* between u and v, and is written $d(u, v)$. (Warning: notation is easily confused with the degree of a vertex.) (This is also called **geodesic distance** or **shortest-path distance**).

Usually we represent all the distances between vertices in a graph as a matrix:



	1	**2**	**3**	**4**	**5**
1	0	3	2	2	1
2	3	0	1	1	2
3	2	1	0	1	1
4	2	1	1	0	1
5	1	2	1	1	0

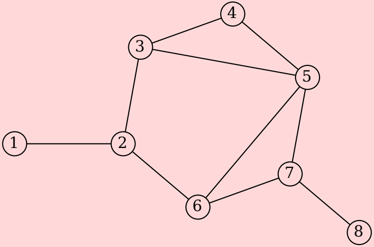
One we have the idea of **distance** we can then define

- **Eccentricity of a vertex:** the greatest distance between that vertex and any other in the graph.
- **Radius of a graph:** the minimum eccentricity of any vertex.
- **Diameter of a graph:** the maximum eccentricity of any vertex. So this is also the maximum entry in the distance matrix.

Example

Consider the following graph.

- Write down the distance matrix for this graph,
- Use the distance matrix to determine the eccentricity of each vertex.
- Determine the radius and diameter of the graph.



MA284
Week 12: Matrices and Review

END OF PART 2

MA284
Week 12: Matrices and Review

Start of ...

PART 3: Additional Topics

Other topics in combinatorics and graph theory that we have not yet covered. The most interesting (to my mind are):

[ie Dr Niall Madden's preference - I might speak about music tomorrow but I'll leave these notes here in case students are interested in the software Niall used to prepare these slides.]

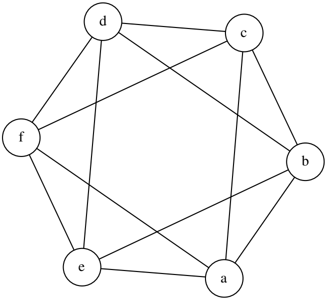
1. Directed Graphs
2. Visualisation of graphs;
3. Algorithms, like determining if a graph is connected, or finding the shortest path between two vertices...
4. The Graph Laplacian;
5. Centrality analysis.
6. and many, many, more...

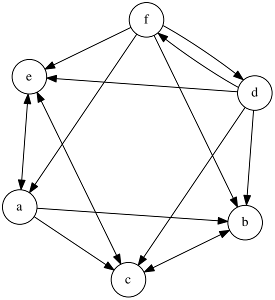
We'll finish now with a short presentation on the first 4 of these.

Graphs often represent networks, such as the road network we had earlier, or social networks. So far, we have had that, if vertex *a* is adjacent to vertex *b*, then *b* is adjacent to *a*.

In many situations, this is not reasonable:

- a city road system might have a one-way system;
- on a social network, you might follow someone who does not follow you back.



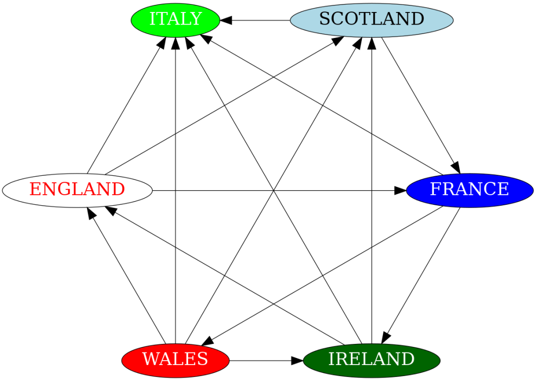


Definition: Directed Graph (=Digraph)

A **directed graph** (also called a "digraph") is one where the edge set is composed of **ordered** pairs, called "**directed edges**" ("arcs"). If $(a, b) \in E$, we say "there is a directed edge from vertex a to vertex b".
When sketching the graph, we indicate the direction of the edge by an arrow.

1. The number of directed edges starting at vertex a is the **out degree** of a.
2. The number of directed edges ending at vertex a is the **in degree** of a.
3. The adjacency matrix is not usually symmetric.

Example: Graph of a tournament (2021 Senior Men's 6 Nations Rugby). A directed edge from Team A to Team B means Team A beat Team B. In this case, the graph is used only to summarise the outcome. However, various algorithms for ranking teams use methods based on graphs.



Because graphs are key to understanding and analysing networks of any type, there are numerous computational tools for working with graphs.

Ones that I have used in the course of this module – mainly for generating images are

- Graphviz, which combines a language called "dot" with a set of tools for converting these graphs to images.
- SageMath, a free computer algebra system, and
- NetworkX, a Python-based system for analysing graphs and networks.

Graphviz

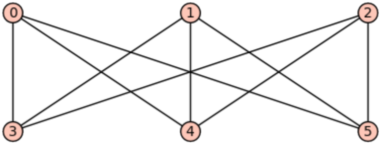
Making K_5

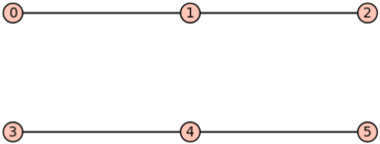
```
graph  {
1 -- {2 3 4 5};
2 -- {3 4 5};
3 -- {4 5};
4 -- {5};
}
```

To generate a image from this you can either install GraphViz on your own computer, or use an online tool such as `http://www.webgraphviz.com/` or `https://dreampuf.github.io/GraphvizOnline`

SageMath

```
G = graphs.CompleteBipartiteGraph(3, 3)
G.show()
H = G.complement()
H.show()
```

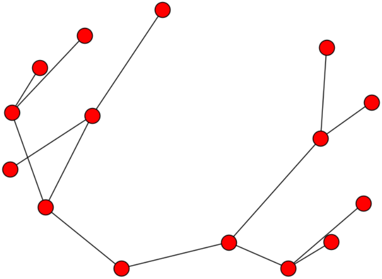




You can do a lot more than this with Sage, including applying numerous algorithms for, say, computing the Chromatic Number of a graph, or finding a minimum spanning tree.

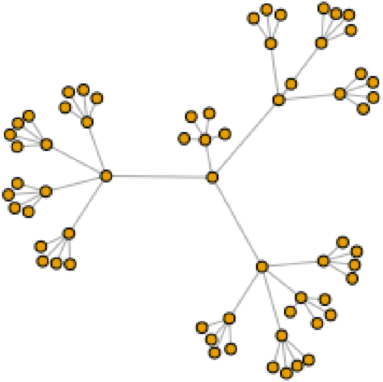
NetworkX

```
import matplotlib.pyplot as plt
import networkx as nx
G = nx.balanced_tree(2,3)
nx.draw(G), plt.show()
```



R with the igraph library

```
library(igraph)
tree <- make_tree(64, 4, mode="undirected")
plot(tree, vertex.size=6, vertex.label=NA)
```



MA284
Week 12: Matrices and Review

END OF PART 3

MA284
Week 12: Matrices and Review

Start of ...

PART 4: Review

Th set of topics that we studied includes:

1. The additive and multiplicative principles;
2. Sets; the Principle of **Inclusion/Exclusion** (PIE) and its applications;
3. **Binomial Coefficients** (& lattice paths, bit-strings, & Pascal's triangle);
4. Permutations and Combinations;
5. **Stars and Bars**, and the NNI Equations and Inequalities;
6. Algebraic V **Combinatorial** Proofs;
7. Derangements;
8. Counting functions;

9. Graph Theory: motivation and basic definitions;
10. Isomorphisms between graphs.
11. Important families of graphs (Cycle graphs, K_n, $K_{n,n}$, etc.)
12. Planar & non-planar graphs; chromatic numbers, **Euler's formula**,
13. Convex polyhedra, and Platonic solids;
14. Graph Colouring; Greedy and Welsh-Powell algorithms;
15. Eulerian and Hamiltonian graphs;
16. Trees, including spanning trees, and decision trees.
17. Matrices of Graphs.

There are **8** questions on the final MA284 exam: you should attempt *all* eight. 4 questions are worth 13 marks, and 4 are worth 12.

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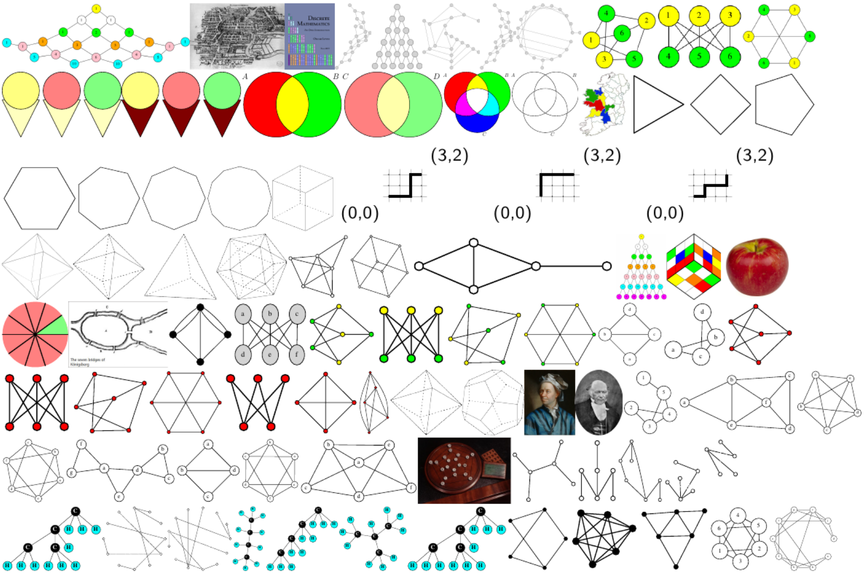
Tips:

- The questions on the exam are roughly in the order in which we covered the topics in class.
- 4 questions are on combinatorics, and 4 are on graph theory.
- The Principles of Addition, Multiplication, and Inclusion/Exclusion are essential to most of the combinatorics questions.
- Good idea to review the homework exercises.
- For graph theory, you need to know how to
  - sketch a graph given the edge and vertex sets;
  - determine if the graph is, e.g., bipartite, planar, connected, ...
  - find an Eulerian path/circuit.
  - compute the chromatic number
  - calculate the radius and diameter.

A summary in one slide...



I'm especially grateful to Drs Niall Madden in particular and Angela Carnevale who pitched a very interesting cross-section of discrete maths as a syllabus and who prepared such wonderful slides.

Good luck with your exams and I hope you've enjoyed studying this introduction to discrete maths. We have plenty more graph theory in some of our final year modules eg Networks and in some Applied Maths modules eg Modelling.

Q1. Write down the adjacency for each of the following graphs.

(a) $G = (V, E)$ with with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.
(b) K_5
(c) C_5
(d) $K_{3,3}$

Q2. Determine if the following matrices represent adjacency matrices of simple connected graphs. If not, explain why.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Q3. Write down the distance matrix for the followingt graph, and use it to determine the eccentricity of each vertex. Determine the radius and diameter of the graph.