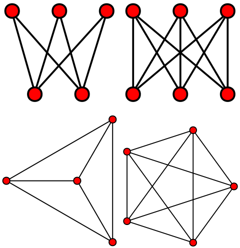
MA284 : Discrete Mathematics

# Week 9: Convex Polyhedra

Dr Kevin Jennings

## 2nd and 4th November, 2022





See also Section 4.3 of Levin's *Discrete Mathematics.*

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# PART 1: Non-Planar graphs

### Recall: Planar graph

- If you can sketch a graph so that none of its edges cross, then it is a *PLANAR* graph.
- When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a *FACE*. The "exterior" of the graph is considered a face.

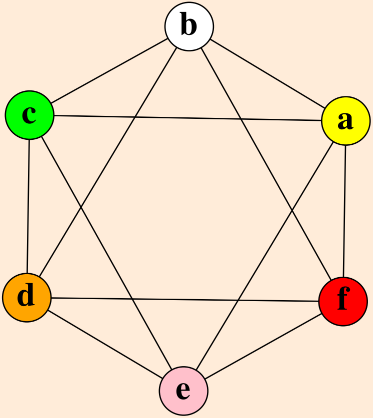
### Euler's formula for planar graphs

For any (connected) planar graph with $v$ vertices, $e$ edges and $f$ faces, we have

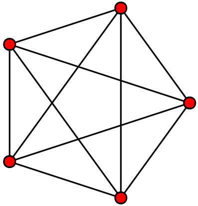
$$v - e + f = 2$$

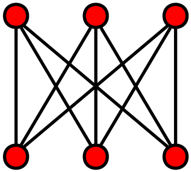
**Example**

Give a planar representation of the following graph, and verify that Euler's Formula Holds.



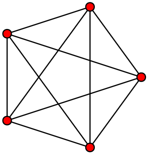
Of course, most graphs do **not** have a planar representation. We have already met two that (we think) cannot be drawn so no edges cross: $K_5$ and $K_{3,3}$:

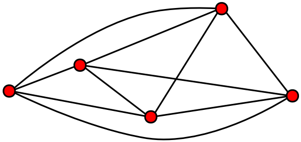




However, it takes a little work to *prove* that these are non-planar. While, through trial and error, we can convince ourselves these graphs are not planar, a proof is still required.

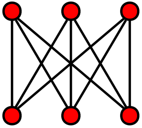
For this, we can use **Euler's formula for planar graphs** to *prove* they are not planar.

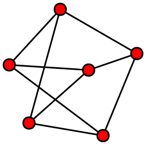




**Theorem (Theorem 4.3.1 in textbook)**

*$K_5$ is not planar. (The proof is by contradiction).*





**Theorem ($K_{3,3}$ is not planar)**

*This is Theorem 4.2.2 in the text-book. Please read the proof there.*

The proof for $K_{3,3}$ is somewhat similar to that for $K_5$, but also uses the fact that a bipartite graph has no 3-edge cycles.

This also means we have solved (negatively) the Utilities (Water-Power-Gas) problem from Week 7.

The understand the importance of $K_5$ and $K_{3,3}$, we first need the concept of *homeomorphic* graphs.

Recall that a graph $G_1$ is a *subgraph* of $G$ if it can be obtained by deleting some vertices and/or edges of $G$.

A *SUBDIVISION* of an edge is obtained by "adding" a new vertex of degree 2 to the middle of the edge.

A *SUBDIVISION* of a graph is obtained by subdividing one or more of its edges.

**Example:**

Closely related: **SMOOTHING** of the pair of edges $\{a, b\}$ and $\{b, c\}$, where $b$ is a vertex of degree 2, means to remove these two edges, and add $\{a, c\}$.

**Example:**

The graphs $G_1$ and $G_2$ are *HOMEOMORPHIC* if there is some subdivision of $G_1$ is isomorphic to some subdivision of $G_2$.

**Examples:**

There is a *celebrated* theorem due to Kazimierz Kuratowski. *The proof is beyond what we can cover in this module. But if you are interested in Mathematics, read up in it: it really is a fascinating result.*

**Theorem (Kuratowski's theorem)**

*A graph is planar if and only if it does not contain a subgraph that is homeomorphic to $K_5$ or $K_{3,3}$.*

What this *really* means is that *every* non-planar graph has some smoothing that contains a copy of $K_5$ or $K_{3,3}$ somewhere inside it.

**Example**

The Petersen graph is not planar `https://upload.wikimedia.org/wikipedia/commons/0/0d/Kuratowski.gif`

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**END OF PART 1**

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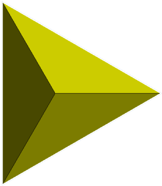
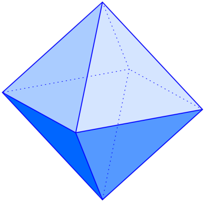
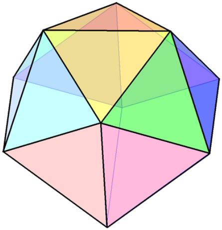
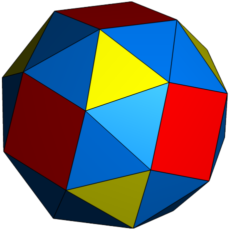
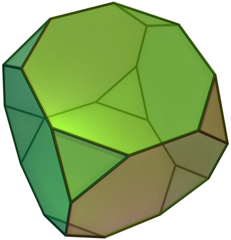
# PART 2: Polyhedra

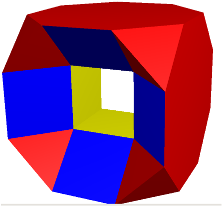




A *polyhedron* is a geometric solid made up of flat polygonal faces joined at edges and vertices.
A *convex polyhedron*, is one where any line segment connecting two points on the interior of the polyhedron must be entirely contained inside the polyhedron.

**Examples:**



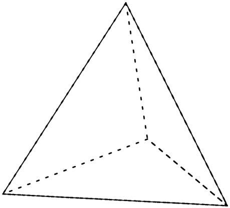




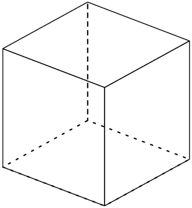
Source: WikiMedia Uniform polyhedron-43-s012.png, Truncatedhexahedron.jpg and Excavated_truncated_cube.png

A remarkable, and important fact, is that *every* convex polyhedron can be projected onto the plane without edges crossing.

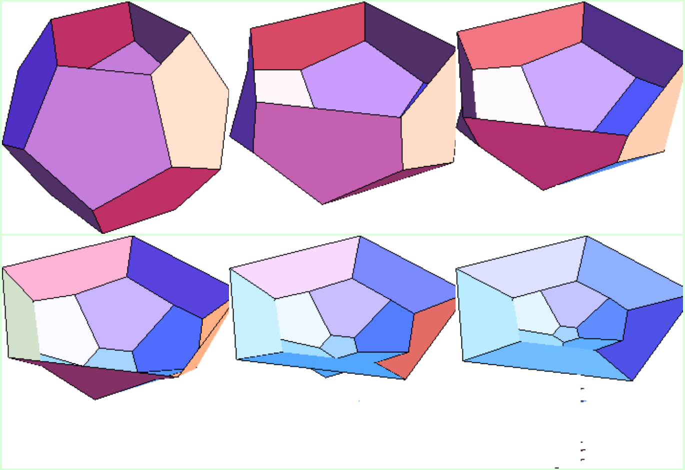
**Example:**



**Example:**

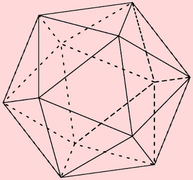


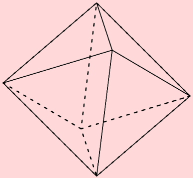
## Example: the dodecahedron



## Exercise

Give a planar projection of each of the following polyhedra.





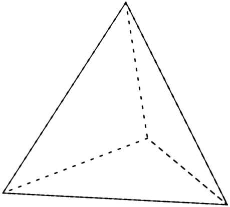
Now that we know every convex polyhedron can be represented as a planar graph, we can apply Euler's formula.

**Euler's formula for polyhedra**

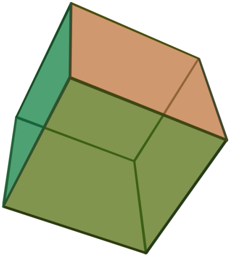
If a convex polyhedron has $v$ vertices, $e$ edges and $f$ faces, then

$$v - e + f = 2$$

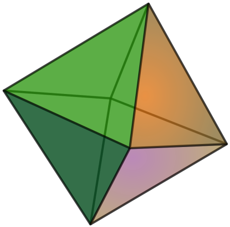
**Example: the tetrahedron.**



**Example: the cube**



**Example: the octahedron**



We now have two very powerful tools for studying convex polyhedra:

- **Euler's formula:** If a convex polyhedron has $v$ vertices, $e$ edges and $f$ faces, then $v - e + f = 2$
- (The Handshaking Lemma) **The sum of the vertex degrees is $2|E|$:** let $G = (V, E)$ be a graph, with vertices $V = v_1, v_2, \ldots, v_n$. Let $\deg(v_i)$ be the *"degree of $v_i$"*. Then

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|.$$

**Example (See textbook, Section 4.2 (Polyhedra))**

Show that there is no convex polyhedron with 11 vertices, all of degree 3?

**See textbook, Example 4.2.3**

Show that there is no convex polyhedron consisting of

- 3 triangles,
- 6 pentagons, and
- 5 heptagons (7-sided polygons).

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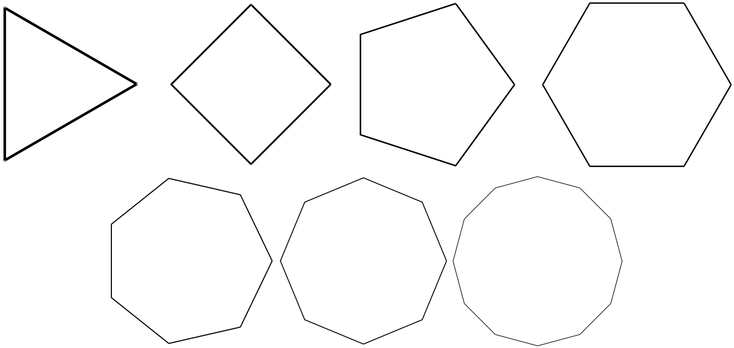
# PART 3: Platonic Solids

*Regular polyhedra - they are surprisingly few of them!*

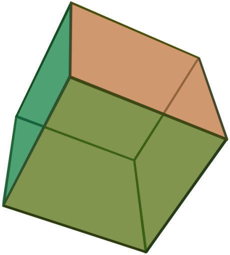
A **POLYGON** is a two-dimensional object. It is *regular* if all its sides are the same length:

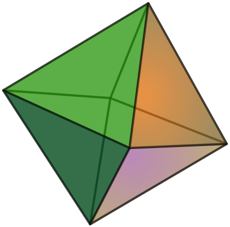


A *polyhedron* with the following properties is called **REGULAR** if

- All its faces are identical regular polygons.
- All its vertices have the same degree.

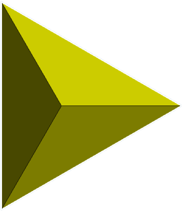
The convex regular polyhedra are also called *Platonic Solids*. Examples:

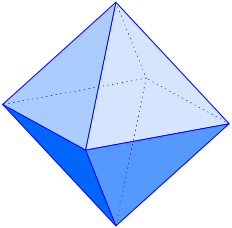


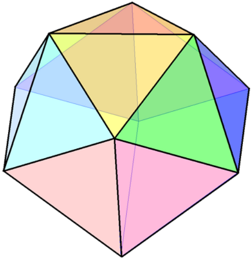


## There are exactly 5 regular polyhedra

This fact can be proven using Euler's formula.
For full details, see the proof in the text book.
Here is the basic idea: we will only look at the case of polyhedra with triangular faces.







[All images here courtesy of Wikipedia]

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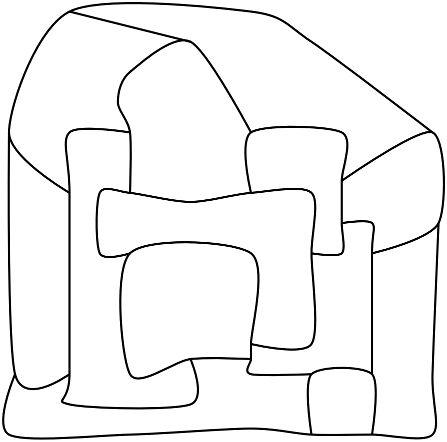
# PART 4: Vertex Colouring

[From textbook, p184]. Here is a map of the (fictional) country "Euleria". Colour it so that adjacent regions are coloured differently. What is the fewest colours required?



There are maps that can be coloured with

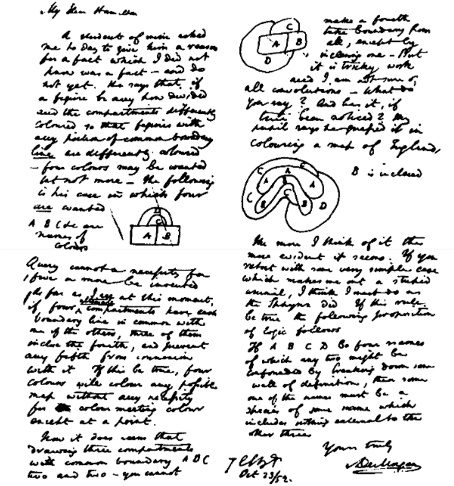
- A single colour:
- Two colours (e.g., the island of Ireland):
- Three colours:
- **Four colours:**

It turns out that the is *no* map that needs more than 4 colours. This is the famous Four Colour Theorem, which was originally conjectured by the British/South African mathematician and botanist, Francis Guthrie who at the time was a student at University College London.

He told one of his mathematics lecturers, Augustus de Morgan, who, on **23 October, 1852**, wrote to friend William Rowan Hamilton, who was in Dublin:

**From** `https://en.wikipedia.org/wiki/Four_color_theorem`

de Morgan writes to Hamilton, 23 October, 1852..



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary ABC two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly

ADeMorgan

7 Camden St

Oct 23/52.

**From `https://en.wikipedia.org/wiki/Four_color_theorem`**
de Morgan writes to Hamilton, 23 October, 1852..

*A student of mine asked me to day to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured-four colours may be wanted, but not more-the following is his case in which four are wanted. Query: cannot a necessity for five or more be invented... What do you say? And has it, if true been noticed?*

*My pupil says he guessed it in colouring a map of England... The more I think of it, the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did...*

De Morgan needn't have worried: a proof was not produced until **1976**. It is very complicated, and relies heavily on computer power.

To get a sense of *why* it might be true, try to draw a map that needs 5 colours.

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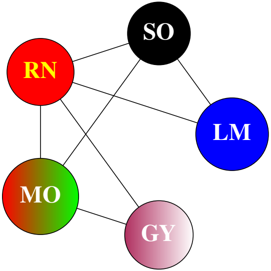
Our interest is not in trying to prove the Four Colour Theorem, but in how it is related to Graph Theory.

If we think of a map as a way of showing which regions share borders, then we can represent it as a *graph*, where

- A vertex in the graph corresponds to a region in the map;
- There is an edge between two vertices in the graph if the corresponds regions share a border.

**Example:**



Colouring regions of a map corresponds to colouring vertices of the graph. Since neighbouring regions in the map must have different colours, so too adjacent vertices in the graph must have different colours.

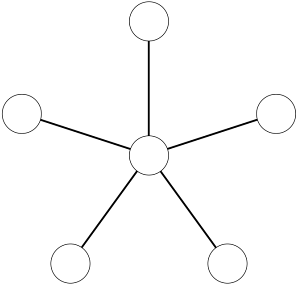
More precisely

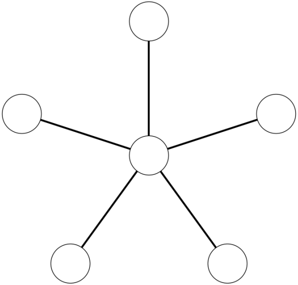
Vertex Colouring: An assignment of colours to the vertices of a graph is called a *VERTEX COLOURING*.

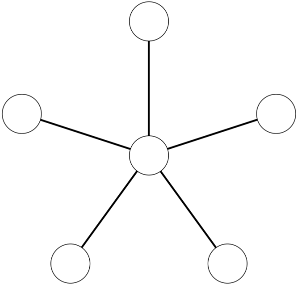
Proper Colouring: If the vertex colouring has the property that adjacent vertices are coloured differently, then the colouring is called *PROPER*.

Lots different proper colourings are possible. If the graph as $v$ vertices, the clearly at most $v$ colours are needed. However, usually, we need far fewer.

**Examples:**







## CHROMATIC NUMBER

The smallest number of colours needed to get a proper vertex colouring if a graph $G$ is called the *CHROMATIC NUMBER* of the graph, written $\chi(G)$.

**Example:** Determine the Chromatic Number of the graphs $C_2$, $C_3$, $C_4$ and $C_5$.

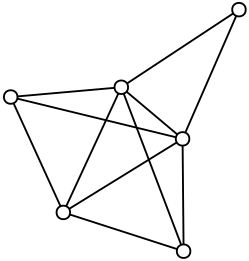
**Example:** Determine the Chromatic Number of the $K_n$ and $K_{p,q}$ for any $n$, $p$, $q$.

In general, calculating $\chi(G)$ is not easy. There are some ideas that can help. For example, it is clearly true that, if a graph has $v$ vertices, then

$$1 \leq \chi(G) \leq v.$$

If the graph happens to be *complete*, then $\chi(G) = v$. If it is *not* complete the we can look at *cliques* in the graph.

Clique: A *CLIQUE* is a subgraph of a graph all of whose vertices are connected to each other.



The **CLIQUE NUMBER** of a graph, $G$, is the number of vertices in the largest clique in $G$.

From the last example, we can deduce that

**LOWER BOUND**: The chromatic number of a graph is *at least* its clique number.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

We can also get a useful upper bound. Let $\Delta(G)$ denote the largest degree of any vertex in the graph, $G$,

**UPPER BOUND**: $\chi(G) \leq \Delta(G) + 1$.

*Why?* This is called **Brooks' Theorem**, and is Thm 4.5.5 in the text-book: `http://discrete.openmathbooks.org/dmoi3/sec_coloring.html`

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Most of these questions are taken from Levin's *Discrete Mathematics*.

Q1. Try to prove that $K_{3,3}$ is non-planar using *exactly* the same reasoning as that used to prove $K_5$ is non-planar. What does wrong? (The purpose of this exercise is to show that noting that $K_{3,3}$ has no 3-cycles is key. Also, we want to know that $K_5$ and $K_{3,3}$ are non-planar for different reasons).

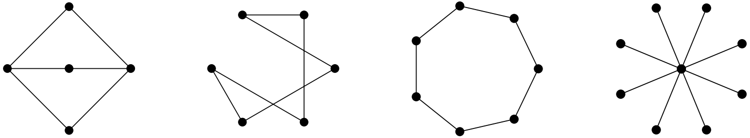
Q2. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

Q3. The graph G has 6 vertices with degrees 2,2,3,4,4,5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

Q4. Euler's formula ($v - e + f = 2$) holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has $k$ components?

Q5. Prove that any planar graph with $v$ vertices and $e$ edges satisfies $e \leq 3v - 6$.

Q6. Which of the graphs below are bipartite? Justify your answers.



Q7. For which $n \geq 3$ is the graph $C_n$ bipartite?

Q8. For each of the following, try to give two different unlabeled graphs with the given properties, or explain why doing so is impossible.

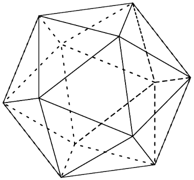
(a) Two different trees with the same number of vertices and the same number of edges. (A tree is a connected graph with no cycles).

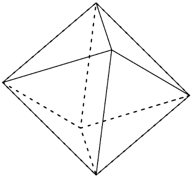
(b) Two different graphs with 8 vertices all of degree 2.

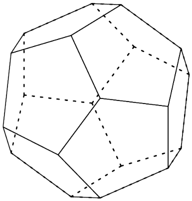
(c) Two different graphs with 5 vertices all of degree 4.

(d) Two different graphs with 5 vertices all of degree 3.

Q9. Give a planar projection of each of the following polyhedra.







Q10. Show that there is only one regular convex polygon with square faces.

Q11. Show that there is only one regular convex polygon with pentagonal faces.

Q12. Could there be a regular polygon with faces that have more than 5 sides? Explain your answer.