Week 8
Lect. 15

For A $n \times n$ matrix

$$\underbrace{\dim(\ker A)}_{\text{nullity}} + \operatorname{rank} A = n$$

Eg Recall the matrix $A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -3 & 4 & -3 \\ 3 & -5 & 5 & -4 \\ -1 & 1 & -3 & 2 \end{pmatrix} \longrightarrow$

row operations $\longrightarrow$ (see earlier lectures) $\begin{pmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so clearly rows 1 & 2 are linearly indep.

$\Rightarrow \operatorname{rank} A = 2$

and solving $AX = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ & & & & 0 \end{array}\right)$

the only pivots are in rows 1 & 2.
in x_1 & x_2 positions (columns) (non-free variables)

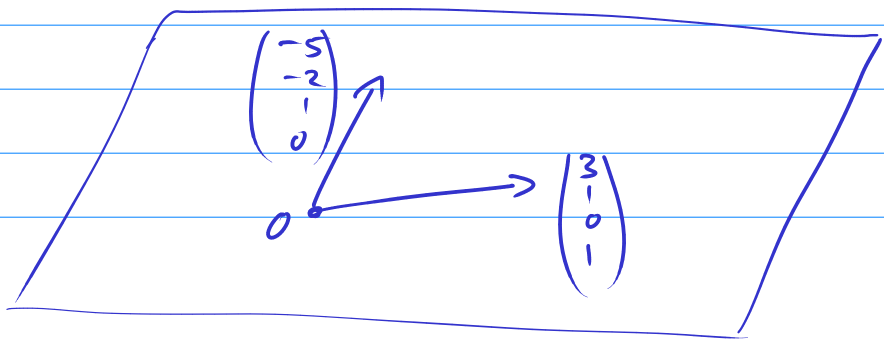
$\Rightarrow$ x_3 and x_4 are free and can assume any real nos t & s as values respectively

Row 2 $\Rightarrow$ $x_2 + 2x_3 - x_4 = 0 \Rightarrow x_2 = -2x_3 + x_4$
$\Rightarrow x_2 = -2t + s$

Row 1 $\Rightarrow$ $x_1 + 5x_3 - 3x_4 = 0 \Rightarrow x_1 = -5t + 3s$

$$\Rightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5t+3s \\ -2t+s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

ie a 2-dimensional sol^n space to $AX = 0$



and again
$\operatorname{rank} A + \dim(\ker A) = 4$
$2 + 2 = 4$

§ Inverses via determinants

Recall: For a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\det A \text{ or } |A| \text{ or } \Delta := a_{11}a_{22} - a_{12}a_{21}$$

and if $\det A \neq 0$

then $A^{-1} = \dfrac{1}{|A|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

Exercise: Verify that $AA^{-1} = I$

Now: If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow[R_3 \times a_{11}]{R_2 \times a_{11}} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{pmatrix}$

$$\xrightarrow[R_3 \to R_3 - a_{31}R_1]{R_2 \to R_2 - a_{21}R_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ 0 & a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} \end{pmatrix}$$

If A is invertible either the $(2,2)$ entry or the $(3,2)$ entry or both are nonzero.
Now multiply row 3 by $(a_{11}a_{22} - a_{21}a_{12})$ & then to this new row 3 add $-(a_{11}a_{32} - a_{12}a_{31})$ times row 2 to get

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}|A| \end{pmatrix}$$

$\boxed{\text{check!}}$

where $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{31}$
$\qquad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

which ~~we want~~ must be nonzero if A^{-1} exists.

Recall that for a 3×3 matrix A its determinant $\det A$ or $|A|$ we defined as given in terms of determinants of 2×2 matrices as

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

where A_{ij} is the 2×2 matrix obtained from A by deleting the i^{th} row & j^{th} column.

We can similarly inductively define the det of an $n\times n$ matrix A.

Eg If A was a 4×4 matrix define $\det A$ by:

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - a_{14}\det A_{14}$$

where now the A_{ij} are 3×3 matrices

So define the det of a 5×5 as

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - a_{14}\det A_{14} + a_{15}\det A_{15}$$

In general (next day)

Eg. $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

$$\det A = 1.\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= 1\left(4(0) - -1(-2)\right) - 5\underbrace{\left(2(0) - (-1)(0)\right)}_{0} + 0\left(\quad\right)$$

$$= 1\left(1(-2)\right)$$

$$= -2$$