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Lect 3: Dot Product & hyperplanes

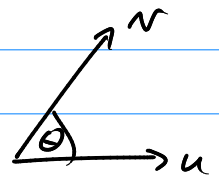
defⁿ: Let $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$
The norm (or length) of v denoted $\|v\|$
is defined to be

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Properties of the Dot Product

- i) $u \cdot v = v \cdot u \quad \forall u, v \in \mathbb{R}^n$ " \forall = for every "
- ii) $u \cdot (v+w) = u \cdot v + u \cdot w \quad \forall u, v, w \in \mathbb{R}^n$
- iii) $(ru) \cdot v = u \cdot (rv) = r(u \cdot v) \quad \forall u, v \in \mathbb{R}^n, \forall r \in \mathbb{R}$
- iv) In \mathbb{R}^2 we will prove that
 $u \cdot v = \|u\| \|v\| \cos \theta$

where θ is the angle between u & v .



As in higher dimensions any 2 vectors define a 2-dimensional plane; we also have the same notion of angle θ .

Corollary: $u \cdot v = 0 \iff \cos \theta = 0$
 $\implies \theta = \pi/2$

So u & v are perpendicular or orthogonal.

defⁿ: The hyperplane in \mathbb{R}^n with normal
 $n = (n_1, n_2, \dots, n_n)$ through the origin $O = (0, 0, \dots, 0)$
is the set of vectors (or points)
 $x = (x_1, x_2, \dots, x_n)$ that are perpendicular to n
(orthogonal)

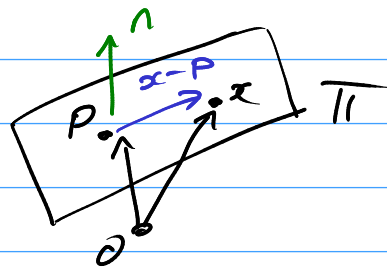
$$\text{So } n \cdot x = n_1 x_1 + n_2 x_2 + \dots + n_n x_n = 0$$

When $n=3$ we just call it the plane.. instead of hyperplane.

Ex If the plane goes through some other point, say $P = (p_1, p_2, p_3)$ & again has normal $n = (n_1, n_2, n_3) \neq 0$

Then the equation of the plane is

$$\Pi: \{ (x_1, x_2, x_3) = x \mid (x-P) \cdot n = 0 \}$$



$x-P \perp n$.

So the eqn is

$$x \cdot n - P \cdot n = 0$$

i.e. $x \cdot n = P \cdot n := d \in \mathbb{R}$

i.e. $n_1 x_1 + n_2 x_2 + n_3 x_3 = d$

Ex Find the eqn of the plane in \mathbb{R}^3 with normal $n = (2, 3, 1)$ passing through the point $P = (1, 1, 2)$.

By the above, if $x = (x_1, x_2, x_3)$ lies on the plane then $n \cdot x = n \cdot P \Rightarrow$

$$(2, 3, 1) \cdot (x_1, x_2, x_3) = (2, 3, 1) \cdot (1, 1, 2)$$

$$\Rightarrow 2x_1 + 3x_2 + x_3 = 2 + 3 + 2$$

i.e. $2x_1 + 3x_2 + x_3 = 7$

Check that $P = (1, 1, 2)$ lies on this

Ans: $2(1) + 3(1) + 2 \stackrel{??}{=} 7$

$$2 + 3 + 2 = 7 \quad \checkmark \quad \text{So yes.}$$

It is intuitively clear that there is a unique plane Π through 3 points (in general position) in \mathbb{R}^3 . How to find the eqn of Π .

Ex: Find the eqn of the plane Π in \mathbb{R}^3 containing $(1,1,2)$, $(-3,4,6)$ and $(0,5,7)$.

The eqn is of the form
 $n_1x_1 + n_2x_2 + n_3x_3 = d$

$$\begin{aligned} \Rightarrow \quad n_1 + n_2 + 2n_3 &= d_1 & \textcircled{1} & \quad (1,1,2) \in \Pi \\ -3n_1 + 4n_2 + 6n_3 &= d_2 & \textcircled{2} & \quad (-3,4,6) \in \Pi \\ 5n_2 + 7n_3 &= d_3 & \textcircled{3} & \end{aligned}$$

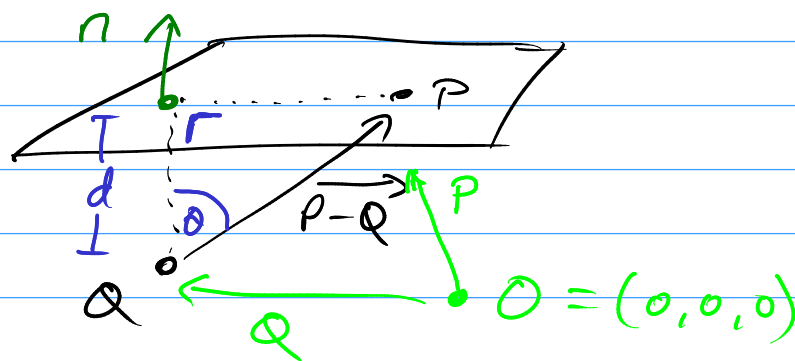
So we need to solve 3 linear eqns in the 3 unknowns n_1, n_2 & n_3 (where d_1, d_2, d_3 are given real numbers)

We expect that this is just enough to yield a unique solⁿ

eg last year we saw that generally 2 eqns in 2 unknowns have a unique solⁿ, namely the intersection point of the 2 lines in \mathbb{R}^2 with the given eqn.

Geometrically (generally) 3 planes in \mathbb{R}^3 intersect in just one point (2 intersect in a line & this line meets the 3rd plane in a unique point)

Find the distance d from a point Q to the plane π with normal n & containing P . We assume $Q \notin \pi$



$$d = \|P-Q\| \cos \theta$$

$$\text{Also } (P-Q) \cdot n = \|P-Q\| \|n\| \cos \theta \\ = d \|n\|$$

$$\text{Thus } d = \frac{|(P-Q) \cdot n|}{\|n\|}$$

• Solving Systems of Linear Equations and Echelon Form

Some systems are easily solved eg

$$\left. \begin{array}{l} x_1 + 2x_2 + x_3 = 16 \quad (i) \\ x_2 + 2x_3 = 10 \quad (ii) \\ 3x_3 = 9 \quad (iii) \end{array} \right\} \text{system}$$

This is because of the "upper diagonal" shape of the system of eqns, i.e. eqn (ii) involves one less unknown than eqn (i) and eqn (iii) involves one less eqn than eqn (ii)

Solⁿ:

A solution to the above system of linear eqns

(i.e. more than one eqn & the unknowns x_1, x_2, x_3 can only be multiplied by reals and then added or subtracted)

is just a vector (c_1, c_2, c_3)

so that if $x_1 = c_1, x_2 = c_2, x_3 = c_3$ then all three eqns (i), (ii) & (iii) are satisfied.

In this case,

$$(iii) \Rightarrow x_3 = 3. \text{ Back substituting into (ii)}$$

$$(ii) \Rightarrow x_2 + 2(3) = 10 \Rightarrow x_2 = 4.$$

& finally back substituting into (i)

$$i) \Rightarrow x_1 + 2(4) + 3 = 16 \Rightarrow x_1 = 5$$

So in this case we have a unique solution $x_1 = 5, x_2 = 4$ & $x_3 = 3$ or simply $(5, 4, 3)$.

Aside: We will also use the definition of matrix multiplication to write a system of linear eqns as one matrix eqn.

Above let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

3x1

$$\& b = \begin{pmatrix} 16 \\ 10 \\ 9 \end{pmatrix}$$

3x1

3x3
rows

Then our system can be written as

$$Ax = b$$

$(3 \times 3)(3 \times 1) = (3 \times 1)$ sizes

Idea: Change a given system of linear eqns into a system that has a "simple" form as above and that has the same solutions i.e. to a so called "equivalent system".

Question: What can we do to a system of linear eqns without changing the solutions?

Elementary Operations:

Ans: (1) Interchange 2 equations

(2) Multiply an eqn by a nonzero real number

(3) Replace an equation by itself plus a multiple (non-zero) of some other eqn in the system.

Weds (25) When we write the system of equations in 'Extended Matrix Notation'

$$\begin{array}{l} \text{i.e.} \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad (i) \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad (ii) \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \quad (n) \end{array}$$

Rewrite as $Ax = b$
or simply encode this info in the
extended (or augmented) matrix

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{array} \right)$$

so that equation

- (i) corresponds to Row 1 of $(A|b)$
- (ii) corresponds to Row 2 of $(A|b)$
- ⋮
- (n) " " Row n " "

Letting R_i denote Row i of $(A|b)$

Then the Elementary (Row) Operations
are written as:

- 1) Interchange R_i & R_j
- 2) Replace R_i by kR_i , $k \neq 0$, $k \in \mathbb{R}$
- 3) Replace R_i by $R_i + kR_j$, $j \neq i$, $k \in \mathbb{R}$

Ex Consider the system of 3 linear eqns
in 3 unknowns x_1, x_2 & x_3 .

$$\left. \begin{array}{l} 2x_1 + 3x_2 + 2x_3 = 100 \\ x_1 + x_2 + 4x_3 = 70 \\ 20x_1 + 10x_2 + 10x_3 = 500 \end{array} \right\} \Leftrightarrow \left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array} \right) \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} \\ = \left(\begin{array}{c} 100 \\ 70 \\ 500 \end{array} \right)$$

Instead, keep track in the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array} \right) := (A|b)$$

Now use the Elementary Row Operations so that $(A|b)$ is in so called "echelon form" i.e. an "upper triangular" form like the first "simple" eqs we solved last lecture.

The idea is to use the first nonzero entry in a row (pivot) to place zeros in the column entries below.

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 2 & 3 & 2 & 100 \\ 20 & 10 & 10 & 500 \end{array} \right)$$

pivot

$$\begin{array}{l} R_1 \text{ fixed} \\ \longrightarrow \\ R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 20R_1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & -10 & -70 & -900 \end{array} \right)$$

pivot (fluke that this is 1)

$$\begin{array}{l} R_1 \text{ fix} \\ \longrightarrow \\ R_2 \text{ fix (pivot)} \\ R_3 \rightarrow R_3 + 10R_2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array} \right)$$

So Row 3 says $-130x_3 = -1300$
 $\Rightarrow x_3 = 10$

Row 2 says: $x_2 - 6x_3 = -40$

$\Rightarrow x_2 - 6(10) = -40 \Rightarrow x_2 = 20$

& Row 1 says: $x_1 + x_2 + 4x_3 = 70$

$\Rightarrow x_1 + 20 + 4(10) = 70$

$\Rightarrow x_1 = 10$

defn: We say that a system of linear eqns is

- consistent if it has a solution
ie either a unique solⁿ or infinitely many
- inconsistent otherwise ie if it has no solⁿ.

Ex: The above system has a unique solution, so is consistent.

Ex Find the solⁿs of, if any, of

$$\left. \begin{array}{l} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{array} \right\}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array} \right)$$

$R_1 \leftrightarrow R_2$

~~\rightarrow~~ $\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array} \right)$

$$R_1, R_2 \text{ fixed} \rightarrow \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -1\frac{1}{2} \end{pmatrix}$$

$R_3 \rightarrow R_3 - \frac{5}{2}R_1$

$$\rightarrow \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 2\frac{1}{2} \end{pmatrix}$$

$R_3 \rightarrow R_3 + \frac{1}{2}R_2$

Row 3 says $0x_1 + 0x_2 + 0x_3 = 2\frac{1}{2}$

impossible : INCONSISTENT SYSTEM.

So there is no solution to the system.

Lect 6: Echelon Form & Reduced Echelon Form

Recall in solving the system of eqns with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array} \right)$$

we used elementary row operations to put it in echelon form

$$\xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array} \right)$$

& used back-substitution to get $x_1 = 10, x_2 = 20$ & $x_3 = 10$

Alternatively, we could have continued & placed as many zeros as possible above the pivots in each column as follows

to put the matrix (augmented) into
Reduced Echelon Form

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array} \right) \xrightarrow{R_3 \rightarrow \frac{-R_3}{130}} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & 1 & 10 \end{array} \right)$$

$$\begin{array}{l} R_1 \rightarrow R_1 - 4R_3 \\ R_2 \rightarrow R_2 + 6R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 30 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 10 \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 10 \end{array} \right)$$

$$\begin{aligned} (\Rightarrow) \quad x_1 &= 10 \\ x_2 &= 20 \\ x_3 &= 10 \end{aligned}$$

reduced echelon form

The reduced echelon form is useful when looking for all solutions i.e. the "general solⁿ" of a system with infinitely many solutions

Ex: Find the general solⁿ of the system whose augmented matrix has been reduced to

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right)$$

Nice surprise! The augmented matrix is already in echelon form (the 1st nonzero element (pivot) in each row is to the right of the row above it).

Now put it in reduced echelon form (use pivots to put zeros above them).

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 + 2R_3 \\ R_2 \rightarrow R_2 + R_3 \\ R_3 \text{ fixed} \end{array} \left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right)$$

$$R_1 \rightarrow R_1 - R_2 \rightarrow \begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\begin{aligned} \text{So } x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

The columns with a pivot (a 1st nonzero in a row) are columns 1, 3 & col. 5

↔ non-free variables x_1, x_3 & x_5

The remaining variables x_2 & x_4 are called free variables as they are free to be any real numbers & the non-free variables are determined in terms of the free ones as follows:

$$\begin{aligned} x_1 &= -6x_2 - 3x_4 \Rightarrow x_2 \text{ is free i.e. } x_2 = s \in \mathbb{R} \\ x_3 &= 4x_4 + 5 \Rightarrow x_4 \text{ is free, say } x_4 = t \in \mathbb{R} \\ (x_5 &= 7) \end{aligned}$$

So the "solution space" is the set of all solutions i.e.

$$S = \{ (-6s - 3t, s, 4t + 5, t, 7) \in \mathbb{R}^5 : s, t \in \mathbb{R} \}$$

Exercise: Take special values for s & t eg $s=3, t=2$ & verify that $(-24, 3, 13, 2, 7)$ does indeed satisfy the original eqns.

or let $s=0, t=0$ so that $(0, 0, 5, 0, 7)$ is a solution of $Ax=b$.

Note: The set of all solutions S can also be written as

$$S = \{ (-6s - 3t, s, 4t, t, 0) + (0, 0, 5, 0, 7) \in \mathbb{R}^5 : s, t \in \mathbb{R} \}$$

and (check!)

$$S' = \{ (-6s - 3t, s, 4t, t, 0) \in \mathbb{R}^5 : s, t \in \mathbb{R} \}$$

is the set of all solutions of the system $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; $A = \begin{pmatrix} 1 & 6 & 2 & -5 & -2 \\ 0 & 0 & 2 & -8 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$

called the corresponding HOMOGENEOUS system & $P = (0, 0, 5, 0, 7)$ is one "particular" solⁿ of $Ax = b$ (check!)

This is a general fact for systems of (consistent) linear eqns i.e. the sol^s of $Ax = b$ are all of the form $x' + P$ where x' is a solⁿ of $Ax = 0$ and P is one "particular" solⁿ of $Ax = b$.

Ex: Solve the homogeneous system of eqns:

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 3x_2 + 4x_3 - 3x_4 &= 0 \\ 3x_1 - 5x_2 + 5x_3 - 4x_4 &= 0 \\ -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$$

(L7): Ans: The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ 3 & -5 & 5 & -4 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array} \right) \begin{array}{l} R_1 \text{ fixed} \\ R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 + R_1 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array} \right)$$

$$\begin{array}{l} R_2 \text{ fix} \\ R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 + R_2 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ (echelon form)}$$

$$\begin{array}{l}
 R_1 \rightarrow R_1 + 2R_2 \\
 R_2 \text{ fixed} \rightarrow
 \end{array}
 \left(\begin{array}{ccccc}
 1 & 0 & 5 & -3 & 0 \\
 0 & 1 & 2 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0
 \end{array} \right) \quad \left(\begin{array}{l} \text{reduced} \\ \text{echelon form} \end{array} \right)$$

So the columns with a (nonzero) pivot are Col. 1 & Col. 2 so x_1 & x_2 are non-free (or basic) variables and the others i.e. x_3 & x_4 are free variables i.e. they can be any real numbers r and s .

So the general solⁿ S (or the set of all solutions S) is

$$x_1 = -5x_3 + 3x_4$$

$$x_2 = -2x_3 + x_4$$

$$x_3 = r$$

$$x_4 = s$$

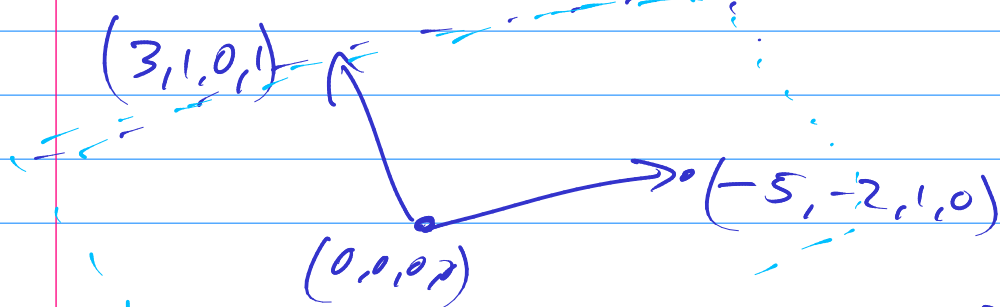
$$\text{i.e. } S = \left\{ (-5r + 3s, -2r + s, r, s) \in \mathbb{R}^4 : r, s \in \mathbb{R} \right\}$$

$$= \left\{ r(-5, -2, 1, 0) + s(3, 1, 0, 1) \in \mathbb{R}^4 : r, s \in \mathbb{R} \right\}$$

Geometrically S is a 2-dimensional plane in \mathbb{R}^4 through the origin $(0, 0, 0, 0)$ because if $x \in S$

$$\text{then } x = (-5r + 3s, -2r + s, r, s)$$

$$= r(-5, -2, 1, 0) + s(3, 1, 0, 1) \text{ \& } r, s \in \mathbb{R}$$



Note: letting $r = s = 0$ we see that $(0, 0, 0, 0) \in S$

§ Linear Independence of vectors in \mathbb{R}^n

We have seen examples of systems of linear eqns which contain redundant equations in the sense that some of the equations may be obtained from others, i.e. in the sense that some of the equations can be written as linear combinations of (the) others.

Equivalently given a collection of vectors in \mathbb{R}^n can we write some of them as a linear combination of the others?

If yes, we say the collection is linearly dependent.

And if not, we say the collection is linearly independent.

Ex: Is the collection
 $v_1 = (1, 2, 1)$, $v_2 = (3, -1, 2)$, $v_3 = (9, 4, 7)$
linearly independent?

Ans: No the vectors are linearly dependent
as $(9, 4, 7) = 3(1, 2, 1) + 2(3, -1, 2)$

$$\Leftrightarrow 1 \cdot (9, 4, 7) - 3(1, 2, 1) - 2(3, -1, 2) = (0, 0, 0)$$

i.e. there exists $x_1, x_2, x_3 \in \mathbb{R}$ not all zero
s.t. $x_1(9, 4, 7) + x_2(1, 2, 1) + x_3(3, -1, 2) = (0, 0, 0)$

(Here $x_1 = 1$, $x_2 = -3$ & $x_3 = -2$)