CS4423: Networks

Week 12, Part 2: Review and Preview

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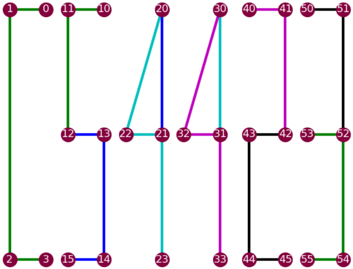


Outline

This weeks notes are split between PDF slides, and a Jupyter Notebook.

Slides are at:

https://www.niallmadden.ie/2425-CS4423

Module Review

2425-CS4423: Some things we learned about in CS4423

- Graph concepts, including order, size, degree, handshaking lemma.
- Important graphs: K_n, $K_{m,n}$, C_n, P_n.
- Adjacency matrices: their powers and their permutations;
- Isomorphic graphs. Line graphs; Graph complements.
- Bipartite graphs, projections, affiliation networks.
- Graphs (and digraphs) as relations.
- Tree and graph traversal

Module Review

- Paths and connection. Connected components.
- (Labelled) Trees, Prüfer codes.
- BFS and DFS, including for distance and shortest paths.
- Distance matrices; radius and diameter
- Centralities:
  - Degree
  - Eigenvector (and Perron-Frobenius theorem)
  - Closeness
  - Betweenness
  - Normalised versions of the above
- Graph Laplacian;
- ER models: $G_{ER}(n, m)$ and $G_{ER}(n, p)$.

Module Review

  - Properties. Expected size. Degree distribution.
  - Appearance of subgraphs. (i.e., triangles).

- Small worlds: Characteristic Path Length; Transitivity and Clustering; Small-world behaviour; Circle graphs and the WS model of random graphs.

- Directed networks; relations; bow-tie structures.

- Lots, and lots of `networkx`

Exam Preview

1. The content is pretty similar to recent exam papers, but not the organisation.
2. Recent exam papers are a good guide, though the Sample Paper is a better one:
   2.1 Unlike recent papers, which had 4 long questions, this one will be like the sample paper: 7 shorter questions.
   2.2 Still have to answer all questions for full marks!
   2.3 The number of marks for each sub-question is listed: these indicate the difficulty and/or effort required.
   2.4 No mention of Bow-Tie diagrams; No explicit mention of spanning trees.

Exam Preview

3. Questions are mostly grouped by topic, and topics are organised in roughly the order in which they came up in class.

4. Sketching graphs is important.

5. There will be graphs given in a variety of formats: figures, edge lists, matrices, `networkx` code, Pruefer code,...

6. The only question that relates to `networkx` will be similar to Q3 on the Sample Paper.

Sample Paper

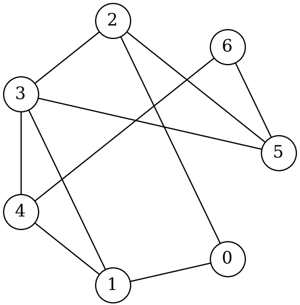
2425-CS4423: Sample Exam Paper

For more information on this paper, and how it relates to the actual example, refer to discussion in Class in Week 12.

Q1(a) Give an example (e.g., by sketching) of a simple connected graph of order 6, and size 7. Is there any simple graph of order 6 and size 16? Explain your answer.

Explain why there is no simple connected graph of order 6 and size 4.

Q1(b) Consider the graph, G_1, shown below. Write down the adjacency matrix, A_1, for G_1.



Q1(c) Explain why G_1 is *not* bipartite.

Give an example of a subgraph of G_1 which is of order 7 and size 8 which *is* bipartite.

Give an example of a subgraph of G_1 which is of order 7 and is a tree.

Q2(a) Sketch the graph with adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Q2(b) Let A be the adjacency matrix of a graph G. Explain how one can compute the size of G as a function of the entries of A.

Q2(c) Let A be the adjacency matrix of a graph G. Explain how one can compute the degree of the nodes of G as a function of the entries of A^2.

Q2(d) Let A be the adjacency matrix of a graph G. Explain how one can compute the number of triangles in G as a function of A^3.

Q3(a) Consider the graph, G_3, generated by the following `networkx` instruction:

```
G3 = nx.Graph([[0,2], [1,2], [2,3], [2,4], [3,4],
    [2,5], [3,4], [3,6], [4,6]])
```

Sketch G_3.

Q3(b) Calculate the *normalised degree centrality* of all nodes in G_3.

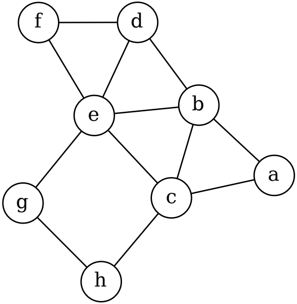
Q3(c) Determine both the radius and diameter of G_3.

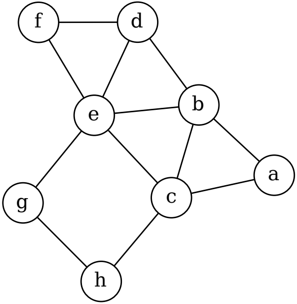
Q3(d) Compute the *closeness centrality* of all nodes in G_3.

Q3(e) If one was to add another edge to G_3. Would that necessarily change both the degree centrality and closeness centrality of some of the nodes in G_3? If so, would they increase or decrease. Explain your answer.

Q4(a) Describe Breadth First Search as an algorithm for computing distances between nodes in a (simple) graph. What is its input, what is its output, and what sequence of steps is taken to produce the output from the input?

Q4(b) Consider the graph, G_4 shown below. Show how to apply the Breadth First Search algorithm, starting at Node a, to determine, for every node, its *predecessors* on the shortest path between it and Node f.
Use this information to list all shortest paths from Node a to Node f.




f
d
e
b
a
g
c
h

Q5(a) Let G_5 be the tree on the nodes $\{0, 1, 2, 3, 4, 5, 6\}$ that has as its *Laplacian matrix*

$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Sketch G_5.

Q5(b) How does one construct the Prüfer code for a tree?
Compute the Prüfer code for G_5 from Part (a).

Q5(c) How does a tree's Prüfer code relate to its degree sequence? Construct the degree sequence for the tree on the nodes $\{0, 1, 2, 3, 4, 5, 6\}$ with Prüfer code $(1, 2, 1, 3, 1)$. Then construct and sketch the tree itself.

Q6(a) m Define the two Erdös-Rényi models, $G_{ER}(n, m)$ and $G_{ER}(n, p)$ of random graphs.

Q6(b) In each model, what is the probability that a randomly chosen graph G has exactly m edges? Justify your answer.

Q6(c) A graph on 120 nodes is constructed by rolling a (fair) 6-sided die once for each possible edge: the edge is added only if the number shown is 3 or 6. What is the probability that a node chosen at random has degree 50? (You do not need to compute a numerical value. It is enough to give an explicit formula in terms of the given data).

Q7(a) What is the *node clustering coefficient* of a node x in a graph G? What is the graph clustering coefficient C of G?

Q7(b) Determine the graph clustering coefficient C of a random graph in the $G_{ER}(n, p)$ model. How does C behave in the limit $n \to \infty$, when the average node degree is kept constant? What practical consequence does this observation have?

Q7(c) Describe the *Watts-Strogatz small-world model* (WS model). What properties does a random graph sampled from the WS model have, that one wouldn't find in a random graph sampled from the $G_{ER}(n, p)$ model, or in an (n, d)-circle graph?