CS4423: Homework Assignment 2: Part 1

Deadline: 5pm, Friday 21 March.

- Your solutions should written by hand and then scanned as a PDF. Then upload to Canvas at `https://universityofgalway.instructure.com/courses/31889/assignments/107917`
- You may collaborate with at most two other people. Each of you must submit your own copy of your work, and that work must include an acknowledgement of who you collaborated with.
- The use of any AI tools, such as ChatGPT or DeepSeek is prohibited, and will be subject to disciplinary procedures.

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Some background: *The Network Laplacian*

Graph Laplacian. There are many ways to represent a network as a matrix, such as the adjacency matrix. Another is the *Laplacian*, $L = (l_{ij})$. For a network $G = (X, E)$ of order n with nodes labelled $1, 2, \dots, n$, L is the square $n \times n$ matrix with entries

$$l_{ij} = \begin{cases} \deg(i) & i = j \\ -1 & \{i, j\} \in X \\ 0 & \text{otherwise} \end{cases}$$

For example, if $G = K_3$, then

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Degree Matrix. The *degree matrix* of a network is the diagonal matrix D with $d_{ii} = \deg(i) = l_{ii}$. So ifG has adjacency matrix A, and degree matrix D, then its Laplacian is

$$L = D - A.$$

Oriented Incidence Matrix. If $G = (X, E)$ has order n and size m, then the **(oriented) incidence matrix**, B, is the $m \times n$ matrix where row i corresponds to the edge $(u_i, v_i) \in X$, and

$$b_{ij} = \begin{cases} 1 & u_i = j \\ -1 & v_i = j \\ 0 & \text{otherwise} \end{cases}$$

Ii is useful to know (but not necessary to prove) that $L = BB^T$.

Q1. Show that, for any network, its Laplacian is singular.

Q2. Explain why we cannot use Perron-Frobenious theory to deduce that the eigenvalue of L with largest modulus (i.e., absolute value) is positive.

Q3. Show that (in fact) all the eigenvalues of L are non-negative.

Q4. Give an example of a network with at least five nodes for which traversal by Depth First Search (DFS) and Breath First Search (BFS) will give the nodes in the same order.

Q5. Let G_1 be the tree with Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Sketch G_1.

Q6. Give the Prüfer code for G_1.

Q7. Sketch the tree, T_2, on the nodes $\{0, 1, \dots, 9\}$ that has as its Prüfer code $[2, 5, 6, 4, 2, 0, 1, 5]$.

Q8. Give the order in which the nodes of T_2 would be visited if it is traversed by **Depth First Searh**.

Q9. Give the order in which the nodes of T_2 would be visited if it is traversed by *Breadth First Searh*.