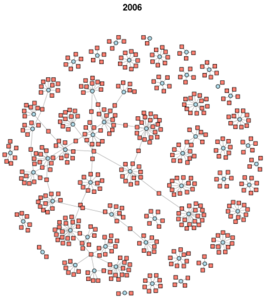
MA284 : Discrete Mathematics

Week 7: Introduction to Graph Theory

Dr Kevin Jennings

19 and 21 October, 2022



See also §1.6, §4.0 and §4.1 of Levin's *Discrete Mathematics: an open introduction*. Some slides are based on ones by Angela Carnevale, all other good stuff due to Niall Madden, mistakes Kevin's.

- **Assignment 1** is closed. Your grade is available on Blackboard.
- **Assignment 2 is open until Thursday afternoon**. You need to access it through Blackboard.
- **Assignment 3** is now open, with a deadline of *5pm, Thursday: 4 November*

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name `WOLLONGONG`.

(ii) How many of these arrangements start with the three `O`'s;

(iii) How many contain the two `G`'s consecutively;

(iv) How many *do not* contain the two `G`'s consecutively?

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PART 1: Counting Functions

(This is actually left over from last week, and not really related to the main topic of the week: Graph Theory)

Recall the $f : A \to B$ is a *function* that maps every element of the set A onto some element of set B. (We call A the "domain", and B the "codomain".)
Each element of A gets mapped to exactly one element of B.

If $f(a) = b$ where $a \in A$ and $b \in B$, we say that "the image of a is b".
Or, equivalently, "b is the image of a".

Examples:

When every element of B is the image of some element of A, we say that the function is *SURJECTIVE* (also called "onto").

Examples:

When no two elements of A have the same image in B, we say that the function is *INJECTIVE* (also called "one-to-one").

Examples:

Bijection

The function $f : A \to B$ is a **BIJECTION** if it is both *surjective* and *injective*. Then f defines a *one-to-one correspondence* between A and B.

Counting functions

Let A and B be finite sets. How many functions $f : A \to B$ are there?

We can use the Multiplicative Principle to deduce:

There are in total $|B|^{|A|}$ functions from A to B.

Counting Bijective Functions (Example 1.3.2 of the textbook)

How many functions $f : \{1, 2, 3, 4, 5, 6, 7, 8\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are bijective?

Remember what it means for a function to be bijective: **each element in the codomain** must be the image of **exactly one element of the domain**. We could write one of these bijections as

What we are really doing is just rearranging the elements of the codomain, so we are defining a **permutation** of 8 elements.

The answer to our question is therefore $8!$.

More generally, there are $n!$ bijections of the set $\{1, 2 \ldots, n\}$ onto itself.

Counting Injective Functions (Example 1.3.2 of the textbook)

How many functions $f\colon \{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ are injective?

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is

$P(8,3) = 8 \cdot 7 \cdot 6.$

Similarly, we can see a k-permutation of $\{1,2,3,\dots,n\}$ as an injective function from $\{1,2,\dots,k\}$ to $\{1,2,3,\dots,n\}$. In general, the number of such injections is $P(n,k)$.

Finally, **derangements** can be interpreted as bijections from a set onto itself and **without fixed points**.

Counting functions without fixed points (see also Section 1.6 of the textbook)

How many **bijective** functions $f\colon \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ are there such that $f(x) \neq x$ for all $x \in \{1,2,3,4,5\}$?

Using our formula

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44.$$

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END OF PART 1

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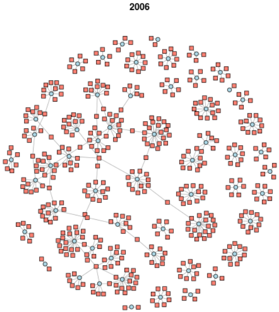
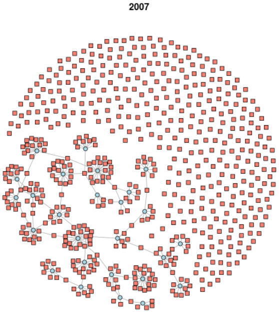
PART 2: Graph Theory

An introduction...

Graph Theory is a branch of mathematics that is several hundred years old. Many of its discoveries were motivated by practical problems, such as determining the smallest number of colours needed to colour a map.

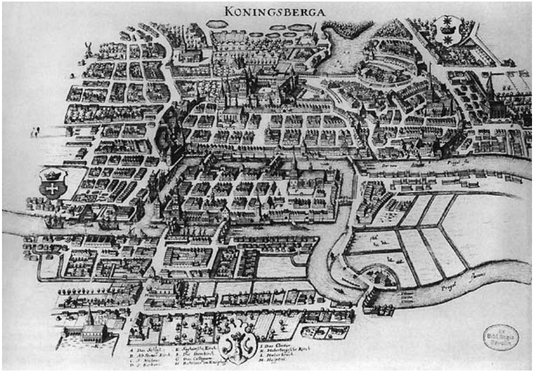
However, it remains one of the most important and exciting areas of modern mathematics, as a bed-rock of data sciences and network theory.





Graph Theory is unusual in that its beginnings can be traced to a precise date.

Königsberg in Prussia (now Kaliningrad, Russia) had seven bridges. Is it possible to walk through the town in such a way that you cross each bridge once and only once?



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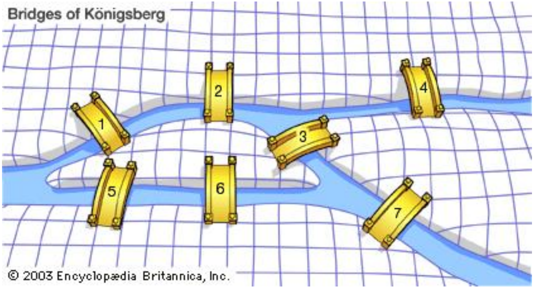

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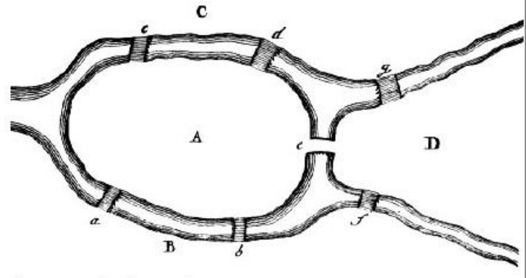
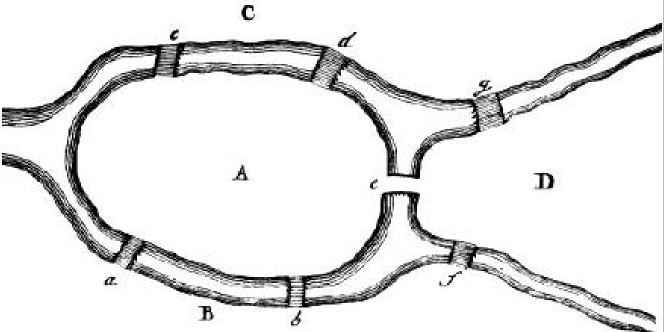


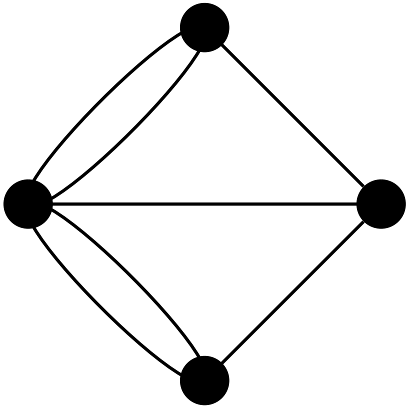
The seven bridges of Königsberg

Is it possible to walk through the town in such a way that you cross each bridge once and only once?



The seven bridges of Königsberg

Here is another way of stating the same problem. Consider the following picture, which shows 4 dots connected by some lines.



Is it possible to trace over each line once and only once (without lifting up your pencil)? You must start and end on one of the dots.

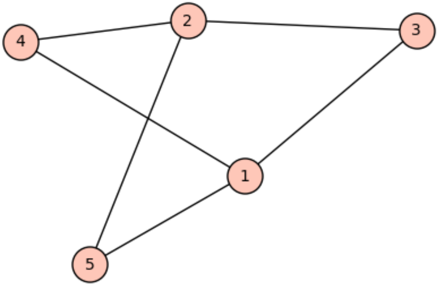
Graph

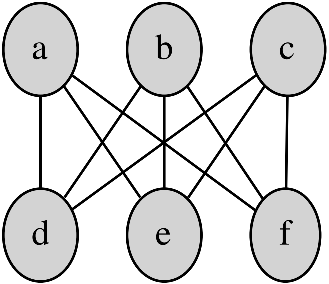
A *GRAPH* is a collection of

- "vertices" (or "nodes"), which are the "dots" in the above diagram.
- "edges" joining pair of vertices.

If the graph is called G (say), we often define it in terms of its *edge set*, E, and *vertex set*, V, as

$$G = (V, E).$$

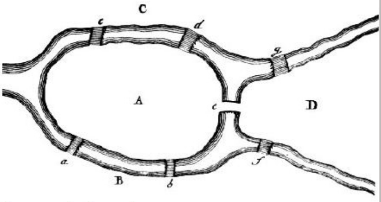




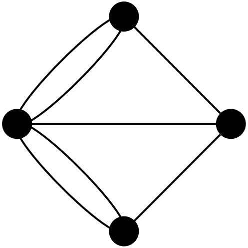
If two vertices are connected by an edge, we say they are *adjacent*.

Graphs are used to represent collections of objects where there is a special relationship between certain pairs of objects.

For example, in the Königsberg problem, the land-masses are vertices, and the edges are bridges.



The seven bridges of Königsberg



(Example 4.0.1 of the text-book)

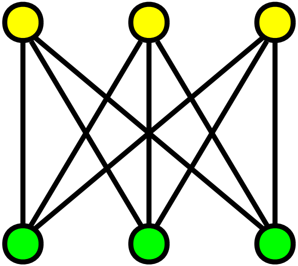
Aoife, Brian, Conor, David and Edel are students in an *Indiscrete Mathematics* module.

- Aoife and Conor worked together on their assignment.
- Brian and David also worked together on their assignment.
- Edel helped everyone with their assignments.

Represent this situation with a graph.

The Three Utilities Problem; also Eg 4.0.2 in text-book

We must make Water, Power and Gas connections to three houses.
Is it possible to do this without the conduits crossing?



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PART 3: Graph Theory - The Basics

Key terms and notation

Definition (ORDER)

The order a graph $G = (V, E)$ is the size of its vertex set, $|V|$.

Let $G = (V, E)$, with

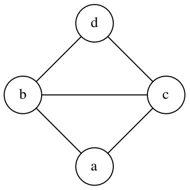
$$V = \{a, b, c, d\}, \qquad E = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\big\}$$

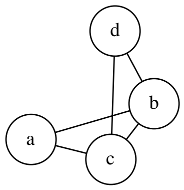
What is the order of G? Sketch G.

Two graphs are *EQUAL* if the have exactly the same Edge and Vertex sets. That is *it is not important how we draw them*, how where we position the vertices, the length of the edges, etc.

Example (Section 4.1 of text-book)

Show that the two graphs given below are *equal*





Isomorphism

An *ISOMORPHISM* between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a *bijection* $f : V_1 \to V_2$ between the vertices in the graph such that, if $\{a, b\}$ is an edge in G_1, then $\{f(a), f(b)\}$ is an edge in G_2.
Two graphs are *ISOMORPHIC* if there is an isomorphism between them. In that case, we write $G_1 \cong G_2$.

Example (Example 4.1.1 of text-book)

Show that the graphs

$$G_1 = \{V_1, E_1\}, \text{ where } V_1 = \{a, b, c\} \text{ and } E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\};$$

$$G_2 = \{V_2, E_2\} \text{ where } V_2 = \{u, v, w\}, \text{ and } E_2 = \{\{u, v\}, \{u, w\}, \{v, w\}\}$$

are not *equal* but are *isomorphic*.

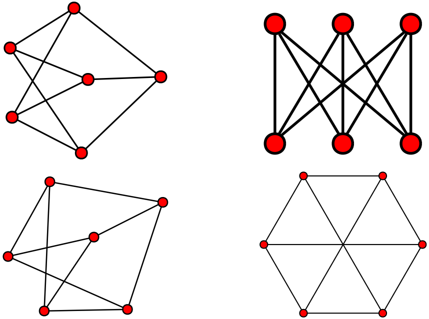
Example (Example 4.1.3 from text-book)

Decide whether the graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are equal or isomorphic, where

$V_1 = \{a, b, c, d\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\}$ and

$V_2 = \{a, b, c, d\}$, $E_2 = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$

When we give a graph without labeling the vertices, we are really talking about *all* graphs that are **isomorphic** to the one we have just drawn. For example, when we draw the following graph, we mean it to represent all those graphs that are isomorphic to the *Water-Power-Gas* graph.



Other than the Königsberg Bridges example, all the graphs we have looked at so far

1. have no *loops* (i.e., no edge from a vertex to itself).
2. have no repeated edges (i.e., there is at most one edge between each pair of vertices).

Such graphs are called *SIMPLE* graphs. But because they are the most common, unless we say otherwise, when we say “graph” we mean “simple graph”.

If a graph does have repeated edges, like in the Königsberg example, we call it a *MULTIGRAPH*. Then the list of edges is not a set, since some elements are repeated: it is a multiset (see Week 5).

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PART 4: Walks, paths, cycles and circuits

Definition (WALK, TRAIL, PATH)

A **WALK** is sequence of vertices such that consecutive vertices are adjacent.
A **TRAIL** is walk in which no edge is repeated.
A **PATH** is a trail in which no vertex is repeated, except possibly the first and last.

Example:

We can also describe a path by the edge sequence. This can be useful, since the **LENGTH** of the path is the number of *edges* in the sequence.

And, since there can be more than one, the **SHORTEST PATH** is particularly important.

Example:

Cycles and Circuits

There are two special types of **path** that we will study later in detail:

Cycle: A path that begins and ends at that same vertex, but no other **vertex** is repeated;

Circuit: A path that begins and ends at that same vertex, and no **edge** is repeated;

These questions are based on exercises in Sections 1.6 and 4.1 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. Consider functions $f : \{1, 2, 3, 4\} \to \{a, b, c, d, e, f\}$. How many functions have the property that $f(1) \neq a$ or $f(2) \neq b$, or both?

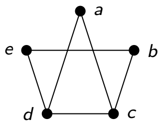
Q2. Consider sets A and B with $|A| = 10$ and $|B| = 5$. How many functions $f : A \to B$ are *surjective*? [*Hint: the answer is* $5^{10} - 5 \times 4^{10} + 10 \times 3^{10} - 10 \times 2^{10} - 5$. *But why?*]

Q3. (Exercise 4.1.1 from text-book) If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

Q4. (Exercise 4.0.2 of text-book *and* MA284 Semester 1 Exam, 2015/2016) Among a group of five people, is it possible for everyone to be friends with exactly <u>two</u> of the other people in the group?
Is it possible for everyone to be friends with exactly <u>three</u> of the other people in the group? Explain your answers carefully.

Q5. Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.

Graph 1: $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, e\}\}$.

Graph 2: 

Q6. (MA284, Semester 1 Exam, 2016/2017) For each of the following pairs of graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, determine if they are isomorphic. If they are, give an isomorphism between them. If not, explain why.

(a) $V_1 = \{a, b, c, d\}$, $\quad E_1 = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\Big\}$ and
$V_2 = \{w, x, y, z\}$, $\quad E_2 = \Big\{\{y, x\}, \{x, z\}, \{z, w\}, \{z, y\}\Big\}$.

(b) $V_1 = \{a, b, c\}$, $\quad E_1 = \Big\{\{a, b\}, \{b, c\}, \{a, c\}\Big\}$ and
$V_2 = \{w, x, y, z\}$, $\quad E_2 = \Big\{\{w, z\}, \{z, y\}, \{w, x\}\Big\}$.

(c) $V_1 = \{a, b, c, d, e\}$, $\quad E_1 = \Big\{\{a, c\}, \{a, e\}, \{b, c\}, \{b, d\}, \{e, c\}, \{d, e\}\Big\}$ and
$V_2 = \{v, w, x, y, z\}$, $\quad E_2 = \Big\{\{v, x\}, \{x, y\}, \{y, z\}, \{z, v\}, \{z, x\}, \{x, w\}\Big\}$.