MA204/MA284 : Discrete Mathematics

Week 6: Advanced PIE, Derangements, and Counting Functions

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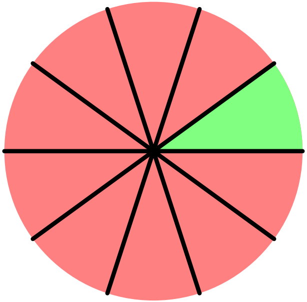
12 and 14 October, 2022



See also §1.6 of Levin's *Discrete Mathematics: an open introduction*. Some slides are based on ones by Dr Angela Carnevale (and Dr Niall Madden of course).

Assignment 2 is now open, with a deadline of *5pm next Tuesday: 18 October*. Like Assignment 1, you must access it through Blackboard.

Assignment 3 opens very soon A practice exercise sheet pdf is already available.

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Start of ...

PART 1: Part 1: Advanced Counting Using PIE

Recall that

- $|X|$ denotes the number of elements in the set X.
- $X \cup Y$ (the *union of X and Y*) is the set of all elements that belong to **either** X or Y.
- $X \cap Y$ (the *intersection of X and Y*) is the set of all elements that belong to *both* X and Y.

The **Principle of Inclusion/Exclusion (PIE)** for two sets, A and B, is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The **Principle of Inclusion/Exclusion (PIE)** for three sets, A, B and C, is

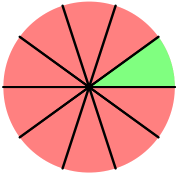
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$$
\begin{aligned}
|A \cup B \cup C \cup D| = &\text{(the sum of the sizes of each single set)} \\
&- \text{(the sum of the sizes of each \textbf{intersection} of 2 sets)} \\
&+ \text{(the sum of the sizes of each \textbf{intersection} of 3 sets)} \\
&- \text{(the sum of the sizes of \textbf{intersection} of all 4 sets)}
\end{aligned}
$$

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



But it is instructive, if a little tedious, to use the PIE to answer this. (See `http://discrete.openmathbooks.org/dmoi3/sec_advPIE.html` for a more detailed solution):

$$\binom{13}{3}-\binom{4}{1}\binom{10}{3}+\binom{4}{2}\binom{7}{3}-\binom{4}{3}\binom{4}{3}+\binom{4}{4}\binom{1}{3}=286-480+210-16=0$$

Not all such problems have easy solution solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...}$$

1. There are no restrictions (other than each x_i being an *nni*).
2. $0 \leq x_i \leq 3$ for each i.

... continued... *2. How many non-negative integer solutions are there to*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...} 0 \leq x_i \leq 3 \text{ for each } i.$$

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END OF PART 1

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PART 2: Derangements

For **Assignment 2** of AM842 (*Indiscreet Mathematics*), students work together in groups of **4**. The group is given a score, which they divide up, according to the amount of work each did, to get their individual scores.
Aoife, Brian, Conor and Dana worked together, and got a score of **10**. They decided it should be divided as:

Aoife: 1 Brian: 2 Conor: 3 Dana: 4.

They informed their lecturer of this, and he tried to enter these on Blackboard. But he is not very good with computers, and got ALL the scores wrong!
How many ways could this happen?

Recall: PERMUTATION

A **permutation** of a collection of objects is a re-ordering of it.

There are $n!$ permutations of a set with n elements.

DEGRANGEMENTS

A **derangement** is a permutation where no item is left in its original place.

Example:

The study of **derangements** dates back to at least 1708. The old French card game called *Rencontres* was a game of chance for two players, A and B:

- The players begin with a shuffled, full deck of 52 cards each.
- Each would take turns placing random cards on the table.
- If any of the cards matched, player A would win.
- If none of the cards matched, player B would win.

In 1708, Pierre de Montort (1678–1719) posed the problem: what is the probability that there would be no matches?

If we let D_{52} be the number of *derangements* of 52 cards then the solution is $D_{52}/52!$.

Let D_n be the number of *derangements* of n objects. First we will work out formulae for D_1, D_2, D_3, and D_4.

In general, the formula for D_n, the number of derangements of n objects is

$$D_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\Big).$$

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END OF PART 2

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PART 3: Counting with repetitions

Suppose we have a set of n objects which are all distinguishable.

(a) How many k-permutations are there (with no repetition)?

(b) How many k-combinations are there (with no repetition)?

But what happens when *some* of the elements of the set are *indistinguishable*?

How many "words" can we make from the following sets of letters?

(i) $\{\texttt{M,A,Y,O}\}$

(ii) $\{\texttt{C,L,A,R,E}\}$

(iii) $\{\texttt{G,A,L,W,A,Y}\}$

(iv) $\{\texttt{R,O,S,C,O,M,M,O,N}\}$

Let's consider the last example carefully: how many "words" can we make from letters in the set $\{\texttt{C},\texttt{M},\texttt{M},\texttt{N},\texttt{O},\texttt{O},\texttt{O},\texttt{R},\texttt{S}\}$?

If somehow the three $\texttt{O}$'s were all distinguishable, and the two $\texttt{M}$'s were distinguishable, the answer would be $9!$.

But, since we can't distinguish identical letters,

- Let's choose which of the 9 positions we place the three $\texttt{O}$'s. This can be done in $\binom{9}{3}$ ways.
- Now let's choose which of the remaining 6 positions we place the two $\texttt{M}$'s. This can be done in $\binom{6}{2}$ ways.
- Now let's choose where to place the remaining 4 letters. This can be done in $4!$ ways.

By the Multiplicative Principle, the answer is

$$\binom{9}{3}\binom{6}{2}4! = \frac{9!}{3!6!}\,\frac{6!}{2!4!}4! = \frac{9!}{3!2!}$$

Multinomial coefficient

The number of different permutations of n objects, where there are n_1 indistinguishable objects of Type 1, n_2 indistinguishable objects of Type 2, ..., and n_k indistinguishable objects of Type k, is

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}$$

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name `WOLLONGONG`.

(ii) How many of these arrangements start with the three `O`'s;

(iii) How many contain the two `G`'s consecutively;

(iv) How many *do not* contain the two `G`'s consecutively?

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END OF PART 3

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PART 4: Counting Functions

Recall the $f : A \to B$ is a *function* that maps every element of the set A onto some element of set B. (We call A the "domain", and B the "codomain".) Each element of A gets mapped to exactly one element of B.

If $f(a) = b$ where $a \in A$ and $b \in B$, we say that "the image of a is b". Or, equivalently, "b is the image of a".

Examples:

When every element of B is the image of some element of A, we say that the function is **surjective** (also called "onto").

Examples:

When no two elements of A have the same image in B, we say that the function is **injective** (also called "one-to-one").

Examples:

Bijection

The function $f : A \to B$ is a **bijection** if it is both *surjective* and *injective*. Then f defines a *one-to-one correspondence* between A and B.

Counting functions

Let A and B be finite sets. How many functions $f : A \to B$ are there?

We can use the Multiplicative Principle to deduce:

There are in total $|B|^{|A|}$ functions from A to B.

Counting Bijective Functions (Example 1.3.2 of the textbook)

How many functions $f\colon \{1,2,3,4,5,6,7,8\} \to \{1,2,3,4,5,6,7,8\}$ are bijective?

Remember what it means for a function to be bijective: **each element in the codomain** must be the image of **exactly one element of the domain**. We could write one of these bijections as

What we are really doing is just rearranging the elements of the codomain, so we are defining a **permutation** of 8 elements.

The answer to our question is therefore 8!.

More in general, there are $n!$ bijections of the set $\{1, 2 \ldots, n\}$ onto itself.

Counting Injective Functions (Example 1.3.2 of the textbook)

How many functions $f: \{1, 2, 3\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are injective?

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is

$P(8, 3) = 8 \cdot 7 \cdot 6.$

Similarly, we can see a k-permutation of $\{1, 2, 3, \ldots, n\}$ as an injective function from $\{1, 2, \ldots, k\}$ to $\{1, 2, 3, \ldots, n\}$. In general, the number of such injections is $P(n, k)$.

Finally, **derangements** can be interpreted as bijections from a set onto itself and **without fixed points**.

Counting functions without fixed points (see also Section 1.6 of the textbook)

How many **bijective** functions $f\colon \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ are there such that $f(x) \neq x$ for all $x \in \{1,2,3,4,5\}$?

Using our formula

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44.$$

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END OF PART 4

Most of these questions are based on exercises in Section 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"

(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q2. (MA284, Semester 1 Exam, 2017/2018) How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

Q3. (MA284, Semester 1 Exam, 2017/2018)

(a) How many non-negative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 < 11,$$

if there are no restrictions?

(b) How many solutions are there to the above problem if $x_2 \geq 3$?

(c) How many solutions are there if each $x_i \leq 4$?

Q4. The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

(a) No present is allowed to end up with its original label? Explain what each term in your answer represents.
(b) Exactly 2 presents keep their original labels? Explain.
(c) Exactly 5 presents keep their original labels? Explain.

Q5. (MA284 Semester 1 Exam, 2018/2019) Let D_n be the number of derangements of n objects. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Q6. (MA284 Semester 1 Exam, 2018/2019) Five students' phones were confiscated on their way into an exam. How many ways can their phones be returned to them so that no one gets their own phone back? How many ways can their phones be returned to them so exactly two people get their own phones back?

Q7. (MA284 Semester 1 Exam, 2015/2016) Give a formula for the number distinct permutations (arrangements) of all the letters in the word `BALLYGOBACKWARDS`.
How many of these begin with an "L"?
How many have all the vowels together?
How many have all the letters in alphabetical order?

Q8. Consider functions $f : \{1, 2, 3, 4\} \to \{a, b, c, d, e, f\}$. How many functions have the property that $f(1) \neq a$ or $f(2) \neq b$, or both?

Q9. Consider sets A and B with $|A| = 10$ and $|B| = 5$. How many functions $f : A \to B$ are *surjective*? [*Hint: the answer is* $5^{10} - 5 \times 4^{10} + 10 \times 3^{10} - 10 \times 2^{10} - 5$. *But why?*]