MA284 : Discrete Mathematics

Week 11: Trees

16 and 18 November, 2022

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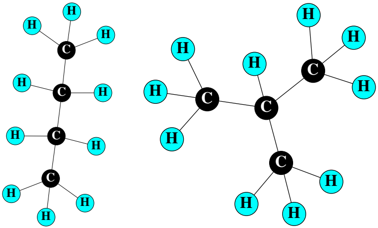


See also Section 4.2 of Levin's *Discrete Mathematics.*

MA284
Week 11: Trees

Start of ...

PART 1: Trees

There's an important class of graphs that do not contain circuits: **TREES**. The mathematical study of trees dates to at least 1857, when Arthur Cayley used them to study certain chemical compounds.

They are used in many mathematical models of decision making (such as Chess programmes), and in designing algorithms for data encoding and transmission.

Definition: ACYCLIC/FOREST

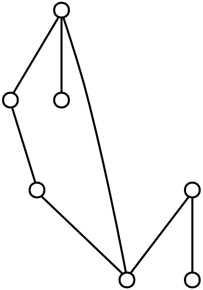
A graph that has no circuits is called **ACYCLIC** or a "forest".

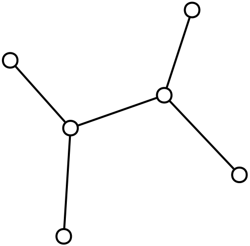
Definition: TREE

A **TREE** is a connected, acyclic graph.

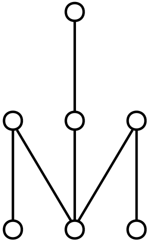
Examples:

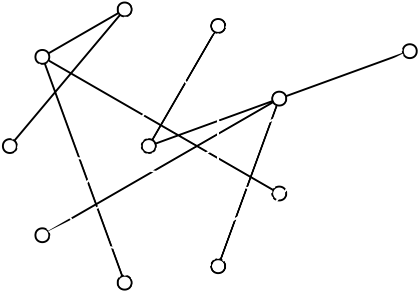
Which of the following are graphs of trees?





Which of the following are graphs of trees?





Another characterisation of trees

A graph is a tree if and only if there is a **unique path** between any two vertices.

If T is a tree, then $e = v - 1$

If T is a tree (i.e., a connected acyclic graph) with v vertices, then it has $v - 1$ edges. (We will see that the converse of this statement is also true).

(See also Prop 4.2.4 in the textbook).

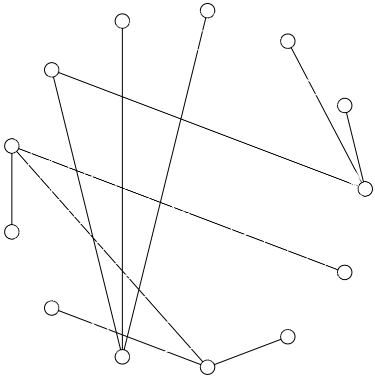
It can be difficult to determine if a very large graph is a tree just by inspection. If we know it has no cycles, then we need to verify that it is connected. The following result (the converse of the previous one) can be useful.

If $e = v - 1$, then T is a tree

If graph with v vertices has *no* cycles, and has $e = v - 1$ edges, then it is a tree.

Example

The following graph has no cycles. Determine how many components it has. Is it a tree?



MA284
Week 11: Trees

END OF PART 1

MA284
Week 11: Trees

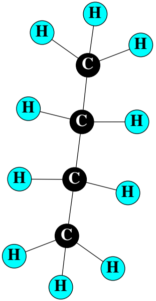
Start of ...

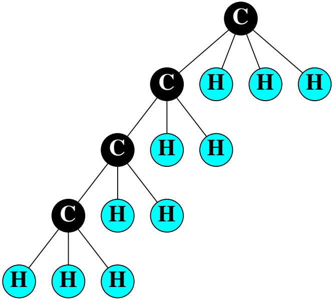
PART 2: Applications

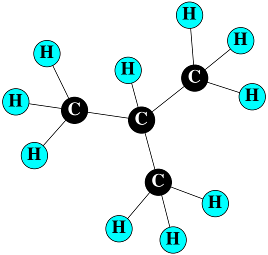
There are many, *many* applications, of trees in mathematics, computer science, and the applied sciences. As already mentioned, the mathematical study of trees began in Chemistry.

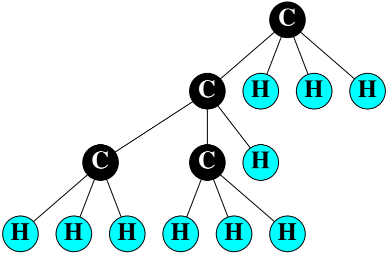
Example: *Saturated hydrocarbon* isomers

Saturated hydrocarbon isomers(alkane) are of the form C_nH_{2n+2}. They have n carbon atoms, and $2n+2$ hydrogen atoms. The carbon atoms can bond with 4 other atoms, and the hydrogens with just one. Show that the graph of all such isomers are trees.









A **DECISION TREE** is a graph where each node represents a possibility, and each branch/edge from that node is a possible outcome.

Example

Pancho and Lefty played a chess match in which there were no drawn games. The first player to win three games in a row or a total of four games won the match.
Pancho won the first game and the person who won the second game also won the third game.
Construct an appropriate tree diagram to find the number of ways in which the match may have proceeded.

(PTO)

Puzzle

You have **eight** identical-looking coins, but one is a counterfeit and lighter than the rest. You have a balance scale. Show that you can find the counterfeit one with just **two** weighing.
How many weighings are needed for **nine** coins? And **ten**?

MA284
Week 11: Trees

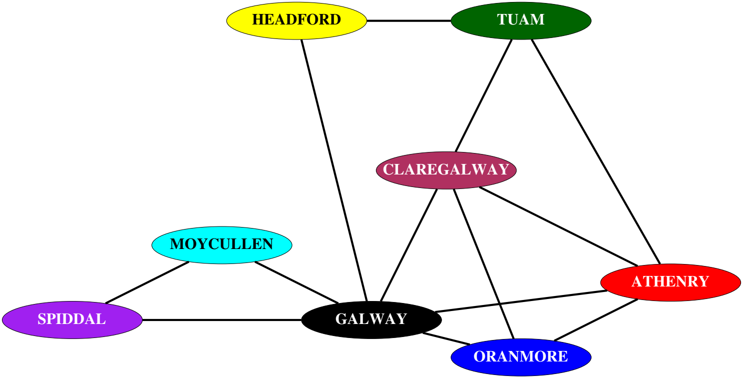
END OF PART 2

MA284
Week 11: Trees

Start of ...

PART 3: Spanning Trees

Consider the road system shown below

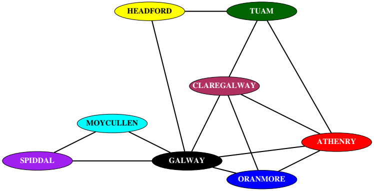


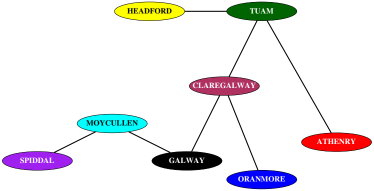
Suppose there has been severe flooding, and Galway County Council can only keep a small number of roads open? Which ones should they choose, so that one can travel between any pair of towns?

Definition: SPANNING TREE

Given a (simple) graph G, a **SPANNING TREE** of G is a subgraph of G that

- is a tree, and
- contains every vertex of G.





Lots of other spanning trees are possible, and there are numerous ways of finding them...

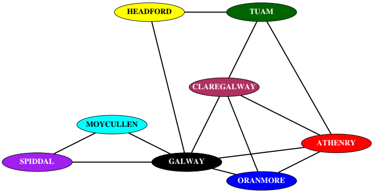
Lots of other spanning trees are possible, and there are numerous ways of finding them. Here are two:

Algorithm 1

(i) Identify a cycle in the graph

(ii) Delete an edge in that cycle, taking care not to disconnect the graph.

(iii) Keep going until all cycles have been removed.

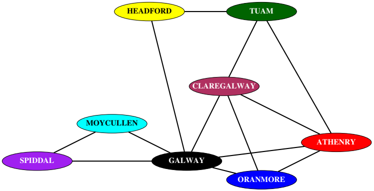


Algorithm 2

(i) Start with just the vertices of the graph (no edges).

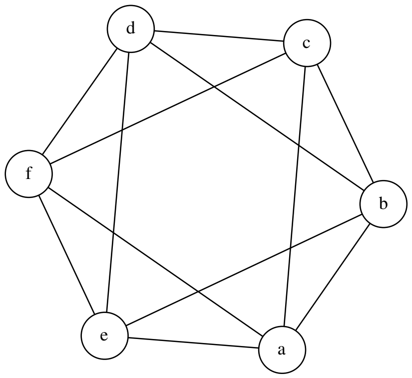
(ii) Add an edge from the original graph, as long as it does not form a cycle.

(iii) Stop when the graph is connected.



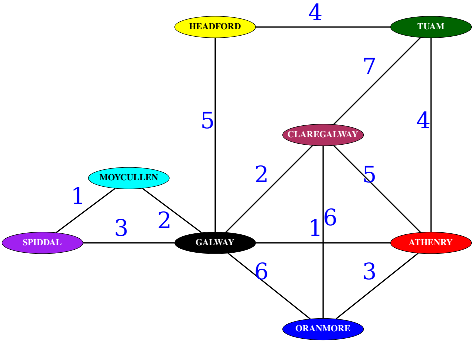
Example

Find a spanning tree in the connected graph illustrated below:



For many applications, we need to consider a wider class of graphs: **weighted graphs**. We won't go into details, but we'll see this with a slight modification of our initial example.

Suppose that one can estimate how long it would take to fix a section of road, as shown by the **weights** on the edges.
Which ones should we fix, so that one can travel between any pair of towns **as soon as possible**?



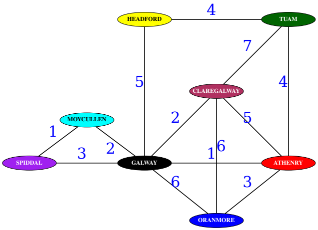
A **minimum spanning tree** is a spanning tree with the minimum possible total edge weight. Minimum spanning trees exist and there are various algorithms to find them.

Algorithm: Kruskal's

(i) Start with just the vertices;

(ii) Add the edge with the least weight that does not form a cycle.

(iii) Keep going until the graph is connected.

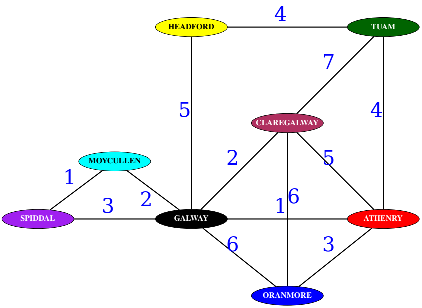


Algorithm: Prim's algorithm

(i) Choose a(ny) vertex from the original graph.

(ii) Add the edge incident to that vertex that has least weight and does not create a cycle.

(iii) Stop when you reached all the vertices of the original graph.



There's a lot of maths involved in planning public transport, roads and all that.

Here's an article on *The Maths of public transport in Galway*, by my colleague Michael Mc Gettrick:

`https://www.rte.ie/brainstorm/2020/0204/1113099-the-maths-of-public-transport/`

There are many other applications of trees that, regrettably, we do not have time to cover. The most important of these include

- minimum spanning trees.
- the study of search algorithms modelled as trees;
- decision tress (like the puzzle from Slide 17);
- compiler syntax;
- Financial modelling: e.g., binomial methods for option pricing;
- The "Good Will Hunting" Problem (draw all homomorphically irreducible trees with $v = 10$ vertices).

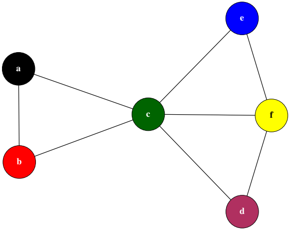
Q1. (See Exer 1 in §4.2 of text). Which of the following graphs are trees?

(a) $G = (V, E)$ with with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.
(b) $G = (V, E)$, with with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.
(c) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}$.
(d) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{d, e\}\}$.

Q2. (See Q2 in Section 4.2 of text-book). For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

(a) $(4, 1, 1, 1, 1)$
(b) $(3, 3, 2, 1, 1)$
(c) $(2, 2, 2, 1, 1)$
(d) $(4, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$.

Q3. Give at least 3 different spanning trees of the graph shown below.



Q4. Give a minimum spanning tree of the weighted graph shown below.