MA284 : Discrete Mathematics

# Week 8: Definitions, and Planar Graphs

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## 26 and 28 October, 2022


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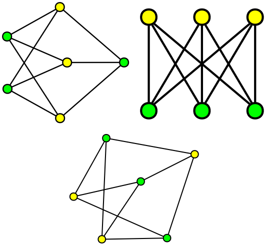

See also Sections 4.1 and 4.3 of Levin's *Discrete Mathematics.*

Some slides are based on ones by Dr Angela Carnevale, most are by Dr Niall Madden, errors are Kevin's.

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# PART 1: Definitions

*An "information dump" on terminology we'll use over the next number of weeks.*

- A *GRAPH* is a collection of
  - "vertices" (or "nodes"), which are the "dots" in the above diagram.
  - "edges" joining pair of vertices.
- A graph is defined in terms of its *edge set* and *vertex set*. That, the graph $G$ with vertex set $V$ and edge set $E$ is written as $G = (V, E)$.
- The **ORDER** of a graph is the number of vertices it has. That is, the order of $G = (V, E)$ is $|V|$.
- If two vertices are connected by an edge, we say they are *adjacent*.
- Two graphs are *EQUAL* if the have exactly the same Edge and Vertex sets.
- An *ISOMORPHISM* between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a *bijection* $f : V_1 \to V_2$ between the vertices in the graph such that, if $\{a, b\}$ is an edge in $G_1$, then $\{f(a), f(b)\}$ is an edge in $G_2$.
- Two graphs are *ISOMORPHIC* if there is an isomorphism between them. In that case, we write $G_1 \cong G_2$.
- A **WALK** is sequence of vertices such that consecutive vertices are adjacent.
- A ***trail*** is a walk in which no edge is repeated is called a trail.
- A ***path*** is a trail in which no vertex is repeated, except possibly the first and last.

### Definition (WALK, TRAIL, PATH)

A **WALK** is sequence of vertices such that consecutive vertices are adjacent.
A **TRAIL** is walk in which no edge is repeated.
A **PATH** is a trail in which no vertex is repeated, except possibly the first and last.

**Example:**

We can also describe a path by the edge sequence. This can be useful, since the **LENGTH** of the path is the number of *edges* in the sequence.

And, since there can be more than one, the **SHORTEST PATH** is particularly important.

**Example:**

## Cycles and Circuits

There are two special types of **path** that we will study later in detail:

Cycle: A path that begins and ends at that same vertex, but no other **vertex** is repeated;

Circuit: A path that begins and ends at that same vertex, and no **edge** is repeated;

A graph is **CONNECTED** if there is a path between every pair of vertices.

**Example:**

The **DEGREE** of a vertex is the number of edges emanating from it. If $v$ is a vertex, we denote its degree as $d(v)$.

If we know the degree of every vertex in the graph then we know the number of edges. This is :

**Lemma (Handshaking Lemma)**

*In any graph, the sum of the degrees of vertices in the graph is always twice the number of edges:*

$$\sum_{v \in V} d(v) = 2|E|.$$

**Example (Application of the Handshaking Lemma)**

Among a group of five people,

(i) is it possible for everyone to be friends with exactly <u>two</u> of the other people in the group?

(ii) is it possible for everyone to be friends with exactly <u>three</u> of the other people in the group?

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# PART 2: Types of Graphs

*Some very important examples of graphs that have special properties*

A graph is **COMPLETE** if every pair of vertices are adjacent. This family of graphs is VERY important. They are denoted $K_n$ – the complete graph on $n$ vertices.

If it is possible to *partition* the vertex set, $V$, into two disjoint sets, $V_1$ and $V_2$, such that there are no edges between any two vertices in the same set, then the graph is **BIPARTITE**.

When the bipartite graph is such that *every* vertex in $V_1$ is connected to *every* vertex in $V_2$ (and *vice versa*) the graph is a **COMPLETE BIPARTITE GRAPH**. If $|V_1| = m$, and $|V_2| = n$, we denote it $K_{m,n}$.

We say that $G_1 = (V_1, E_1)$ is a **SUBGRAPH** of $G_2 = (V_2, E_2)$ provided $V_1 \subset V_2$, and $E_1 \subset E_2$.

We say that $G_1(V_1, E_1)$ is an **INDUCED SUBGRAPH** of $G_2 = (V_2, E_2)$ provided that $V_1 \subset V_2$ and $E_2$ contains *all* edges of $E_1$ which join edges in $V_1$ .

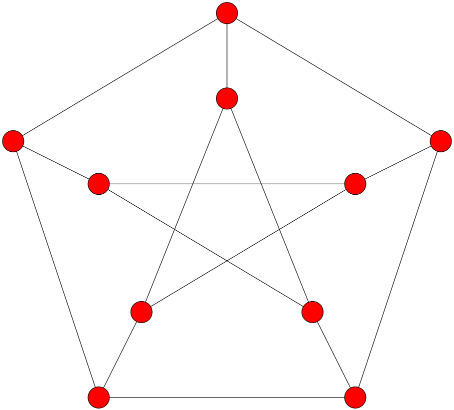
Some graphs are used more than others, and get special names. We already had

- $K_n$ – the complete graph on n vertices.
- $K_{m,n}$ – The complete bipartite graph with sets of m and n vertices.

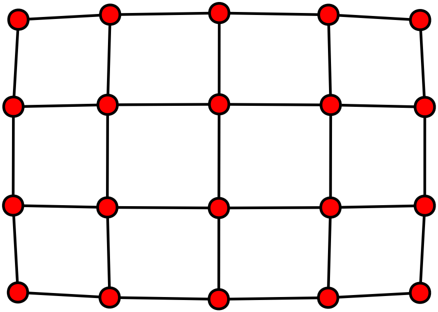
Other important ones include

- $C_n$ – The cycle on $n$ vertices.
- $P_n$ – The path on $n$ vertices.

And there are some graphs that are named after people. The most famous is the *Petersen Graph*.



Two personal favourites are the *Square Grid Graph* and *Triangular Grid Graph*.



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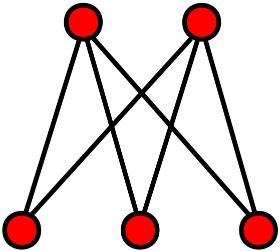
# PART 3: Planar graphs

## Planar graph

If you can sketch a graph so that none of its edges cross, then it is a *planar* graph.

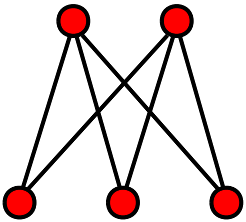
**Example:** The Graph $K_{2,3}$ is *planar:*



These graphs are *equal*. The sketch on the right (see annotated notes) is a *planar representation* of the graph.

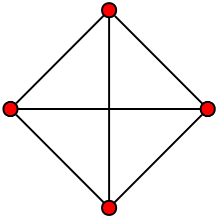
When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions.
Each region is called a *face*.

**Example:** the planar representation of $K_{2,3}$ has **3 faces** (because the "outside" region counts as a face).



The number of faces does not change no matter how you draw the graph, as long as no edges cross.

**Example:** Give a planar representation of $K_4$, and count how many faces it has.



**More examples:** Count the number of edges, faces and vertices in the cycle graphs $C_3$, $C_4$ and $C_5$. What about $C_k$?

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# PART 4: Euler's formula for planar graphs

*Planar graphs are very special in many ways. One of those ways is that there is a relationship between the number of faces, edges and vertices.*

Presently, we'll study a famous formula relating the number of vertices, edges and faces in a *planar* graph.

First, let's try to discover it.

**Example:** Count the number of vertices, edges and faces of $K_{2,4}$.

**Example:** Count the number of vertices, edges and faces of $P_2$, $C_3$, $K_4$, Dickie-bow...

We have produced a list of some planar graphs and counted their vertices, edges, and faces. There is a pattern...

**Euler's formula for planar graphs**

For any (connected) planar graph with $v$ vertices, $e$ edges and $f$ faces, we have

$$v - e + f = 2$$

*Outline of proof:*

(Proof continued).

**Example (Application of Euler's formula)**

Is it possible for a connected planar graph to have 5 vertices, 7 edges and 3 faces? Explain.

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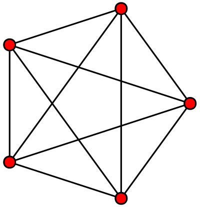
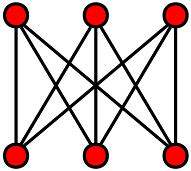
# PART 5: Non-planar graphs

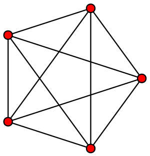
Of course, most graphs do **not** have a planar representation. We have already met two that (we think) cannot be drawn so no edges cross: $K_5$ and $K_{3,3}$:

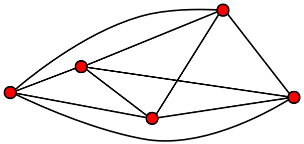




However, it takes a little work to *prove* that these are non-planar. While, through trial and error, we can convince ourselves these graphs are not planar, a proof is still required.

For this, we can use **Euler's formula for planar graphs** to *prove* they are not planar.

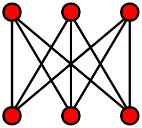
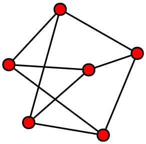




**Theorem (Theorem 4.3.1 in textbook)**

*$K_5$ is not planar.*

The proof is by *contradiction*:

**Theorem ($K_{3,3}$ is not planar)**

*This is Theorem 4.2.2 in the text-book. Please read the proof there.*

The proof for $K_{3,3}$ is somewhat similar to that for $K_5$, but also uses the fact that a bipartite graph has no 3-edge cycles.

This also means we have solved (negatively) the Utilities (Water-Power-Gas) problem from last week.

To understand the importance of $K_5$ and $K_{3,3}$, we first need the concept of *homeomorphic* graphs.

Recall that a graph $G_1$ is a *subgraph* of $G$ if it can be obtained by deleting some vertices and/or edges of $G$.

A *SUBDIVISION* of an edge is obtained by "adding" a new vertex of degree 2 to the middle of the edge.

A *SUBDIVISION* of a graph is obtained by subdividing one or more of its edges.

**Example:**

Closely related: **SMOOTHING** of the pair of edges $\{a, b\}$ and $\{b, c\}$, where $b$ is a vertex of degree 2, means to remove these two edges, and add $\{a, c\}$.

**Example:**

The graphs $G_1$ and $G_2$ are *HOMEOMORPHIC* if there is some subdivision of $G_1$ which is isomorphic to some subdivision of $G_2$.

**Examples:**

There is a *celebrated* theorem due to Kazimierz Kuratowski. *The proof is beyond what we can cover in this module. But if you are interested in Mathematics, read up in it: it really is a fascinating result.*

**Theorem (Kuratowski's theorem)**

*A graph is planar if and only if it does not contain a subgraph that is homeomorphic to $K_5$ or $K_{3,3}$.*

What this *really* means is that *every* non-planar graph has some smoothing that contains a copy of $K_5$ or $K_{3,3}$ somewhere inside it.

**Example**

The Petersen graph is not planar `https://upload.wikimedia.org/wikipedia/commons/0/0d/Kuratowski.gif`

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**END OF PART 5**

Most of these questions are taken from Levin's *Discrete Mathematics*.

Q1. Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

Q2. Try to prove that $K_{3,3}$ is non-planar using *exactly* the same reasoning as that used to prove $K_5$ is non-planar. What does wrong? (The purpose of this exercise is to show that noting that $K_{3,3}$ has no 3-cycles is key. Also, we want to know that $K_5$ and $K_{3,3}$ are non-planar for different reasons).

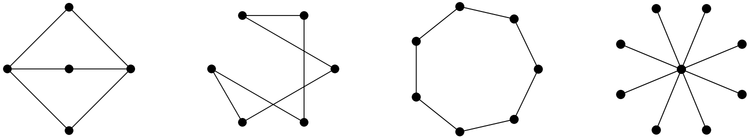
Q3. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

Q4. The graph G has 6 vertices with degrees 2,2,3,4,4,5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

Q5. Euler's formula ($v - e + f = 2$) holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has $k$ components?

Q6. Prove that any planar graph with $v$ vertices and $e$ edges satisfies $e \leq 3v - 6$.

Q7. Which of the graphs below are bipartite? Justify your answers.



Q8. For which $n \geq 3$ is the graph $C_n$ bipartite?

Q9. For each of the following, try to give two different unlabeled graphs with the given properties, or explain why doing so is impossible.

(a) Two different trees with the same number of vertices and the same number of edges. (A tree is a connected graph with no cycles).
(b) Two different graphs with 8 vertices all of degree 2.
(c) Two different graphs with 5 vertices all of degree 4.
(d) Two different graphs with 5 vertices all of degree 3.

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