$$L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longrightarrow L(v)$$

Recap: L linear if

- $L(u+v) = L(u) + L(v) \quad \forall u,v \in \mathbb{R}^n$
- $L(rv) = rL(v) \quad \forall v \in \mathbb{R}^n, \forall r \in \mathbb{R}$

Then $L \longleftrightarrow$ matrix A where $A = \begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \cdots & L(e_n) \\ | & | & & | \end{pmatrix}$

and $L(v) \Longleftrightarrow Av$

is a $m \times n$ matrix (with respect to standard bases in $\mathbb{R}^n$ & $\mathbb{R}^m$)

Observe:

1) L maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$

Because $\underset{\|}{L(0+0)} = L(0)$

$$L(0) + L(0) = 2(0)$$

$$\Rightarrow L(0) = 0$$

Alternatively using the matrix $A \Longleftrightarrow L$

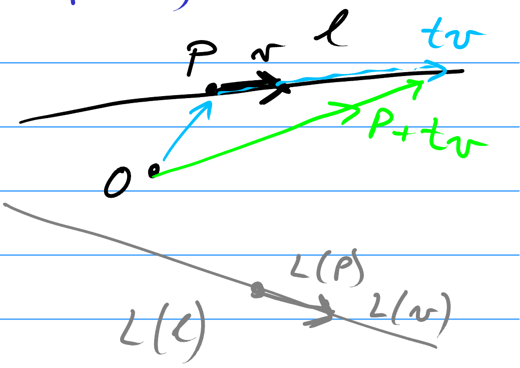
$L(0)$ is obtained as $A \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\in \mathbb{R}^n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m$

* ii) A linear transformation $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ sends lines to lines because given a line ℓ in $\mathbb{R}^n$ (in parametric form)

$$\ell : P + tv, \quad t \in \mathbb{R}$$

then

$$L(\ell) = L(P+tv)$$
$$= L(P) + L(tv)$$
$$= L(P) + tL(v)$$



This is the parametric form
of the line (in $\mathbb{R}^m$)
through the point $L(p)$ in the direction $L(v)$
?? (and $L(v) \neq 0$ if $v \neq 0$)

Examples of Linear Transformations

Ex: Fix a vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$
Define $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as follows:
$L : v \longrightarrow n \times v$ ($\times$ = vector cross product)

Exercise: Check that L is linear ie

$$n \times (v+w) = n \times v + n \times w$$

$$\& \quad n \times (kv) = k(n \times v) \qquad k \in \mathbb{R}$$

Aside: Let $u = (u_1, u_2, u_3)$
$v = (v_1, v_2, v_3)$

Then $u \times v$
"u cross v" $= (u_2v_3 - u_3v_2, \ u_3v_1 - u_1v_3, \ u_1v_2 - u_2v_1)$

Then $u \times v$ is $\perp^r$ to u
& is $\perp^r$ to v (assuming $u \nparallel v$)

Check: $(u \times v) \cdot v$ [should be 0 if they are $\perp^r$]

$$(u \times v) \cdot v = \underline{u_2v_3v_1} - \utilde{u_3v_2v_1} + \utilde{u_3v_1v_2} - u_1v_3v_2 + u_1v_2v_3 - \underline{u_2v_1v_3}$$

$$= 0$$

[Exercise: Check that $(u \times v) \cdot u = 0$ too]

Mnemonic for $u \times v$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$u = u_1e_1 + u_2e_2 + u_3e_3$
$e_1 = (1,0,0)$ etc

determinant (see later) is $e_1(u_2v_3-u_3v_2) - e_2(u_1v_3-u_3v_1)$
$+ e_3(u_1v_2-u_2v_1)$

$= (u_2v_3-u_3v_2 , u_1v_3-u_3v_1 , u_1v_2-u_2v_1)$

End of Aside

Ex ctd To find the matrix A_L for L (wrt the standard basis $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$) we find $L(e_1)$, $L(e_2)$ & $L(e_3)$ and then

$$A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$L(e_1) = n \times e_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= e_1(n_2(0)-n_3(0)) - e_2(n_1(0)-n_3(1)) + e_3(n_1(0)-n_2(1))$$

$$= e_1 \cdot 0 + n_3e_2 - n_2e_3$$

$$= (0, n_3, -n_2)$$

So 1st col of A_L is $\begin{pmatrix} 0 & \vdots & \vdots \\ n_3 & \vdots & \vdots \\ -n_2 & \vdots & \vdots \end{pmatrix}$

Next, $L(e_2) = n \times e_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 1 & 0 \end{vmatrix}$

$$= e_1(n_2(0)-n_3(1)) - e_2\underbrace{(n_1(0)-n_3(0))}_{0} + e_3(1n_1-n_2(0))$$

$$= -n_3e_1 + 0e_2 + n_1e_3$$

$$= (-n_3, 0, n_1)$$

So A_L is $\begin{pmatrix} 0 & -n_3 & \vdots \\ n_3 & 0 & \vdots \\ -n_2 & n_1 & \vdots \end{pmatrix}$ $\leftarrow L(e_3)$

Lect 12:

Finally $L(e_3) = n \times e_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 0 & 1 \end{vmatrix}$

$$= e_1(n_2(1) - n_3(0)) - e_2(n_1(1) - n_3(0)) + e_3(n_1(0) - n_2(0))$$
$$= n_2 e_1 - n_1 e_2 + 0e_3$$
$$= (n_2, -n_1, 0)$$

So 3^{rd} col. of A_L is $\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$

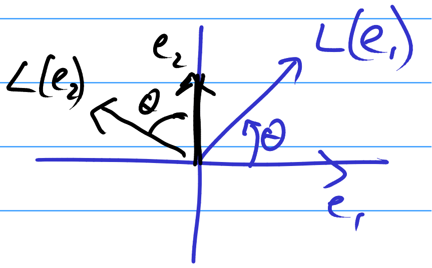
(This is a skew symmetric matrix
ie $A^T = -A$)

Recall from last year the matrix A_L for a rotation L in $\mathbb{R}^2$ about the origin ↺ by an angle θ

We need to find $L(e_1) = L((1,0))$
$L(e_2) = L((0,1))$

and then $A_L = \begin{pmatrix} \uparrow & \uparrow \\ L(e_1) & L(e_2) \\ \downarrow & \downarrow \end{pmatrix}$

We see that (exercise)
$L(e_1) = (\cos\theta, \sin\theta)$
& $L(e_2) = (-\sin\theta, \cos\theta)$



So $A_L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Ex: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the e_3 axis $\curvearrowleft$ by an angle θ.

Then in $e_1 - e_2$ plane we see that

$$L(e_1) = \cos\theta \, e_1 + \sin\theta \, e_2 + 0 e_3$$

& so the first column of A_L is:

$$\begin{pmatrix} \cos\theta & \cdot & \cdot \\ \sin\theta & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix}$$

and $L(e_2) = -\sin\theta \, e_1 + \cos\theta \, e_2 + 0e_3$

so the 2^{nd} col. of A_L is

$$\begin{pmatrix} \cos\theta & -\sin\theta & \cdot \\ \sin\theta & \cos\theta & \cdot \\ 0 & 0 & \cdot \end{pmatrix}$$

and finally

$$\begin{aligned} L(e_3) &= e_3 \\ &= 0e_1 + 0e_2 + 1e_3 \end{aligned}$$

& our 3^{rd} column is

$$\begin{pmatrix} \vdots & \vdots & 0 \\ \vdots & \vdots & 0 \\ \vdots & \vdots & 1 \end{pmatrix}$$

ie

$$A_L = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly the matrix for a rotation about the e_1 axis is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

(check this!)

Ex Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be projection of $\mathbb{R}^3$ onto the x_1-x_2 plane.

ie $L((x_1, x_2, x_3)) = (x_1, x_2)$

So $L((1,0,0)) = (1,0)$
$L((0,1,0)) = (0,1)$
& $L((0,0,1)) = (0,0)$

So $$A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

§ Revision of Matrix Multiplication

Recall that matrix multiplication was so defined so that if

$L_1 \Longleftrightarrow A$ & $L_2 \Longleftrightarrow B$

Then $L_1 \circ L_2 \Longleftrightarrow AB$ ($\neq BA$ usually)

So if $v = (x_1, x_2, \dots, x_n)$ to find $L_1 \circ L_2(v) = L_1(L_2(v))$ we place $v = (x_1, \dots, x_n)$ behind the matrix AB as a column and multiply

$L_2 : \mathbb{R}^n \to \mathbb{R}^m$
&
$L_1 : \mathbb{R}^m \to \mathbb{R}^p$

$$\underset{p\times m}{A}\,\underset{m\times n}{B} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} := \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}_{p\times 1}$$

& $L_1 \circ L_2(v) = (y_1, y_2, \dots, y_p)$

Eg If $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}$ & $B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

Then AB does not make sense but BA does.

Weds: Recall that AB only makes sense if the number of entries in a row of A = the number of entries in a column of B.

i.e. if A is $p \times m$
and B is $m \times n$
Then AB is $p \times \underbrace{m \times m}\times n = p \times n$

If $R = r_1 \ r_2 \ \cdots \ r_m$ is a row of A
& $C = \begin{matrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{matrix}$ is a column of B

Then $RC := r_1c_1 + r_2c_2 + \cdots + r_mc_m$

& we obtain the entries of AB by multiplying all the rows of A by all the columns of B & placing the answer in the corresponding position of AB.

eg i^{th} row of A multiplied by j^{th} col. of B is placed in the i^{th} row & j^{th} col. of AB.

Ex (again) $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

Compute: $A^2 =$ X impossible/undefined
$B^2 =$ ✓ exercise!
$AB =$ X $A.B$ $2 \times \underbrace{3 \times 2}_{\text{no match}} \times 2$
$BA =$

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2(1)+3(3) & 1 & 8 \\ 1(1)+1(3) & 1.(2)+1((-1) & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & 1 & 8 \\ 4 & 1 & 3 \end{pmatrix}$$

eg AB $\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \left(\begin{array}{l} \boxed{\text{Row } 1 \times \text{Col } 1} \\ \uparrow \\ 1.(2) + 2(1) + 1.\boxed{??} \end{array} \right.$

§ Finding the Inverse of a Matrix by Row Operations

Recall that the inverse of an $n \times n$ matrix A is another $n \times n$ matrix denoted A^{-1} s.t.

$$AA^{-1} = A^{-1}A = \underset{\text{"The Identity Matrix"}}{\underset{\downarrow}{I_n}} = \begin{pmatrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{pmatrix}$$

Focus on $\mathbb{R}^3$.

We don't know A^{-1} so let's find its 3 columns X_1, X_2, X_3 as follows.

$$A^{-1} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

so since $AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

⚠ $\Rightarrow AX_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ & $AX_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Ex Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}$

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X_1$ be the 1st col of A^{-1}

We must have that $AX_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ie to find $X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we solve the system

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 2 & 5 & -3 & 0 \\ -3 & 2 & -4 & 0 \end{array}\right) \longrightarrow$$ Put in reduced echelon form to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right)$$

Now to find $X_2 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ we solve $AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

i.e $$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 5 & -3 & 1 \\ -3 & 2 & -4 & 0 \end{array}\right) \longrightarrow$$ Put in Reduced Echelon form to get

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \end{array}\right)$$

⚠

But we just did the same set of row operations again, so:

Do All three at the same time ie

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Row reduce to}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

ie $$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{R_2-2R_1 \\ R_3+3R_1}]{} \begin{pmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 11R_2]{R_1 \to R_1 - 3R_2} \begin{pmatrix} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{exercise}} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \;\; \underbrace{\begin{array}{ccc} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{array}}_{A^{-1}}\right)$$

$\boxed{\begin{array}{l}\text{Fri} \\ \text{L14}\end{array}}$ So what can go wrong so that A^{-1} does not exist?

If A didn't have 3 pivots (positions with leading 1s) we wouldn't be able to reduce to $(I \mid A^{-1})$ e.g.

Let $A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ Then

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 0 & 3 & -5 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 4R_1]{} \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & -9 & 15 & 0 & 4 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 3R_2]{} \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{pmatrix}$$

no 3rd pivot

& clearly inconsistent eqns

Another way to put the problem with $\begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ not having an inverse is that we reduced it to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{pmatrix}$

So that clearly Row 3 = −3 Row 2
ie rows are not linearly independent
or in other words
we were able to find a non-trivial linear combination

$$c_1 R_1 + c_2 R_2 + c_3 R_3 = 0 \quad 0 \quad 0$$

ie the rows are not linearly independent.

The number of linearly independent rows of a matrix doesn't change on performing row operations
but reducing matrix to echelon form makes it easier to spot linearly independent rows.

eg $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ has clearly 2 linearly independent rows (as 1 0 2 can't be a multiple of 0 3 5)

so the original matrix $\begin{pmatrix} 0 & 3 & 5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ has 2 lin. ind. rows

def^n: Let A be an $n \times n$ matrix. The **rank** of A, denoted rank A

$:=$ the no. of linearly indep. rows of A

[Aside: It can be shown that rank A is also equal to the no. of lin. ind. columns of A.]

Theorem: A^{-1} exists (ie A is an invertible matrix) $\iff$ rank $A = n$.

Note if A^{-1} exists and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

then $AX = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$

because
$$AX = 0 \Rightarrow A^{-1}AX = A^{-1}0$$
$$= 0$$
$$\Rightarrow I_n X = 0$$
$$\Rightarrow X = 0$$

On the other hand consider the matrix $A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$

which we have seen has no inverse.

There are $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$

such that $AX = 0$ because

$$\left(\begin{array}{ccc|c} 0 & 3 & -5 & 0 \\ 1 & 0 & 2 & 0 \\ -4 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\text{(exercise!)}]{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

No pivot in row 3
So x_3 is free. Say $x_3 = t \in \mathbb{R}$

$$\Rightarrow \begin{cases} x_1 + 2x_3 = 0 \\ 3x_2 + 5x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} 3x_2 = -5t \\ x_2 = \frac{-5}{3}t \end{array}$$

and $x_1 = -2x_3 \Rightarrow x_1 = -2t$

So the sol^n's of $AX = 0$ are all vectors of the form $X = \begin{pmatrix} -2t \\ \frac{-5}{3}t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ -5/3 \\ 1 \end{pmatrix}$

The vectors X satisfying $AX = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for any matrix A are called the kernel of A.

In this example $\ker A := \{ x \in \mathbb{R}^3 : Ax = 0 \}$
$= \text{span} \left\{ \begin{pmatrix} -2 \\ -5/3 \\ 1 \end{pmatrix} \right\}$

ie it is the line (1 dimensional space) in the direction of the vector $(-2, -5/3, 1)$.

Note: The dimension of the kernel of A + the rank of A $= 3$.

This is true for any $n \times n$ matrix

$$\dim \ker A + \operatorname{rank} A = n.$$