MA284 : Discrete Mathematics

Week 5: Stars and Bars

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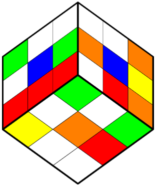
5 and 7 October, 2022



These slides are based on §1.5 of Oscar Levin's *Discrete Mathematics: an open introduction*, and are licensed under CC BY-SA 4.0

ASSIGNMENT 1 should have been closed
ASSIGNMENT 2 is due Oct 18 (extended from Oct 14)
ASSIGNMENT 3 will open soon

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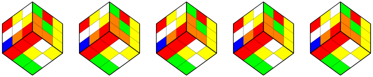
Start of ...

PART 1: Stars and Bars



Think about the following question during this lecture...

Suppose you have some number of identical Rubik's cubes to distribute to your friends. Find the number of different ways you can distribute the cubes...

1. if you have 3 cubes to give to 2 people.
2. if you have 4 cubes to give to 2 people.
3. if you have 5 cubes to give to 2 people.
4. if you have 3 cubes to give to 3 people.
5. if you have 4 cubes to give to 3 people.
6. if you have 5 cubes to give to 3 people.

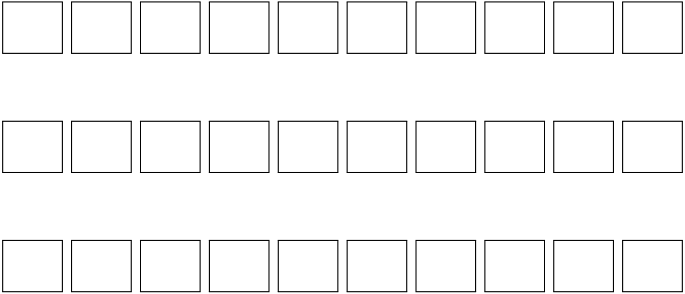
Make a conjecture about how many different ways you could distribute 7 cubes to 4 people. Explain.

What if each person were required to get *at least one* cube? How would your answers change?

Every day you give some apples to your lecturers.
Today you have **7** apples.
How many ways can you give them to **4** lecturers you have today?

Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

One can represent any solution by filling out 10 boxes with 7 stars and 3 bars.
Examples:



Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

- *Every solution can be represented by 10 boxes, each with a star or a bar.*
- There are 7 stars and 3 bars in total.
- We can choose any 3 of the 10 boxes in which to place the bars, and then put the stars in the rest.
- **So we have** $\binom{10}{3}$ **choices for where to put the bars**.

Definition (Multiset)

A ***multiset*** is a set of objects, where each object can appear more than once. As with an ordinary set, order does not matter.

Examples:

How many *multisets* of size 4 can you form using numbers $\{1, 2, 3, 4, 5\}$?

How many *multisets* of size n can you form using the numbers $\{1, 2, 3, \ldots, k\}$?

Example

1. In how many ways can one distribute **ten** €1 coins to four students?
2. In how many ways can one distribute **ten** €1 coins to four students so that each student receives at least €1?

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END OF PART 1

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PART 2: Problems with NNI solutions

In Part 1, we had the following question

How many ways can you share **n** apples among your **k** lecturers?

- This is the same as finding the number of ways we can arrange n apples (stars), divided into k groups, separated by $k-1$ bars.
- Any way can be written with $n+k-1$ symbols (n stars and $k-1$ bars): we just have to choose where to put the $k-1$ bars. This can be done in $\binom{n+k-1}{k-1} = \binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$ ways.
- Each solution can be thought of as a *multiset*: a set of objects, where each object can appear more than once.
- And each can be framed as a solution to the **non-negative integer problem:**

$$x_1 + x_2 + \cdots + x_k = n.$$

All the examples we have looked at so far this week are examples of a broader class of **non-negative integer (NNI) problems**. When we calculate the number of ways of giving 7 apples to 4 lecturers, we are computing the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 7.$$

A non-negative integer (NNI) problem

How many non-negative integer solutions are there to the problem

$$x_1 + x_2 + \cdots + x_k = n?$$

This is the same as...

How many ways are there to distribute n identical objects among k individuals.

The answer is $\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}$

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PART 3: Inequalities

Example (Part 1: Equality)

What are the non-negative integer solutions to

$$x_1 + x_2 + x_3 = 3?$$

Here $n = 3$ and $k = 3$. So we know there are $\binom{5}{2} = 10$ solutions.

They are:

Example (Part 2: Inequality)

How many non-negative integer solutions are there to

$$x_1 + x_2 \leq 3,$$

and what are they?

Example (Part 3: Strict inequality)

How many non-negative integer solutions are there to

$$x_1 + x_2 < 4,$$

and what are they?

In fact, each of the following 3 equations have the same non-negative integer solutions (and, so, same number of solutions):

(1) $x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n.$

(2) $x_1 + x_2 + x_3 + \cdots + x_k \leq n,$

and

(3) $x_1 + x_2 + x_3 + \cdots + x_k < n + 1,$

WHY?

Example

(i) How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

(ii) How many non-negative integer solutions are there of the inequality

$$x_1 + x_2 + x_3 \leq 8$$

Example

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if

1. $x_1 \geq 3$;
2. $x_1 \geq 3$ and $x_2 \geq 3$;
3. Each $x_i \geq 3$

These problems are a little more complicated than the ones we just did.

Example (Upper bounds example 1)

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if:

1. Each $x_i \leq 2$.

Example (Upper bounds example 2)

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if:

2. Each $x_i \leq 3$.

Example (Upper bounds example 2)

How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 = 8$$

if:

3. Each $x_i \leq 4$.

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PART 4: Advanced Counting Using PIE

Recall that

- $|X|$ denotes the number of elements in the set X.
- $X \cup Y$ (the *union of X and Y*) is the set of all elements that belong to **either** X or Y.
- $A \cap B$ (the *intersection of X and Y*) is the set of all elements that belong to *both* X and Y.

The **Principle of Inclusion/Exclusion (PIE)** for two sets, A and B, is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets, A, B and C, the PIE is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$|A \cup B \cup C \cup D| =$ (the sum of the sizes of each single set)

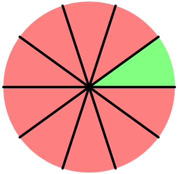
$-$ (the sum of the sizes of each **intersection** of 2 sets)

$+$ (the sum of the sizes of each **intersection** of 3 sets)

$-$ (the sum of the sizes of **intersection** of all 4 sets)

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



Not all such problems have easy solution solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...}$$

1. There are no restrictions (other than each x_i being an *nni*).
2. $0 \leq x_i \leq 3$ for each i.

... continued... *2. How many non-negative integer solutions are there to*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...} 0 \leq x_i \leq 3 \text{ for each } i.$$

Unless indicated otherwise, these questions identical to, or variants on, problems in Section 1.5 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. A *multiset* is a collection of objects, just like a set, but can contain an object more than once (the order of the elements still doesn't matter). For example, $\{1, 1, 2, 5, 5, 7\}$ is a multiset of size 6.

(a) How many *sets* of size 5 can be made using the 10 digits: 0, 1, $\ldots$ 9?
(b) How many *multi*sets of size 5 can be made using the 10 digits: 0, 1, $\ldots$ 9?

Q2. Each of the counting problems below can be solved with stars and bars. For each, say what outcome the diagram $***|*||**|$ represents, if there are the correct number of stars and bars for the problem. Otherwise, say why the diagram does not represent any outcome, and what a correct diagram would look like.

(a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?
(b) How many ways can you distribute 5 identical lollipops to 6 kids?
(c) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$.

Q3. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"

(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q4. (MA284, Semester 1 Exam, 2017/2018) How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

Q5. (MA284, Semester 1 Exam, 2017/2018)

(a) How many non-negative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 < 11,$$

if there are no restrictions?

(b) How many solutions are there to the above problem if $x_2 \geq 3$?

(c) How many solutions are there if each $x_i \leq 4$?