6. Some Discrete Probability Distributions: the Binomial and Poisson

1

1

Learning outcomes

- Describe the Binomial distribution and identify when it is applicable
- Calculate Binomial probabilities
- Describe the Poisson distribution and identify when it is applicable
- Calculate Poisson probabilities

2

2

Links between descriptive stats and probability theory

Data	Random variable
x1, x2, … xn	X
Empirical distributions (plots of relative frequencies)	Pmf, pdf
Sample mean	E(X)
Sample variance	Var(X)
Sample sd	Sd(X)

3

3

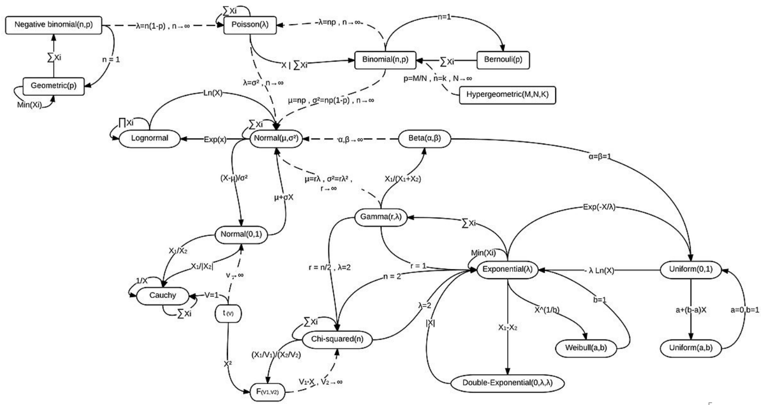
Motivation

- Often, the observations generated by different statistical experiments have the same general type of behaviour.

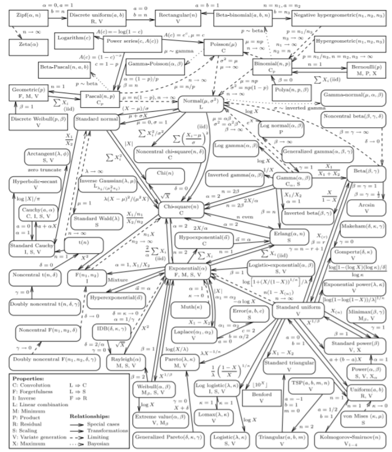
- In general only a handful of important probability distributions are needed to describe many of the discrete random variables encountered in practice.

4

4



5



6

Binary Outcomes

Bernoulli Trial:
Random experiment with just two outcomes — **success/failure**
heads/tails; yes/no; death/survival; ...

7

Binary Outcomes

Bernoulli Trial:
Random experiment with just two outcomes — **success/failure**
heads/tails; yes/no; death/survival; ...

For a single trial, random variable

$$X = \begin{cases} 1 & \textit{success} \\ 0 & \textit{failure} \end{cases}$$

8

Binary Outcomes

Bernoulli Trial:
Random experiment with just two outcomes — **success/failure**
heads/tails; yes/no; death/survival; ...

For a single trial, random variable

$$X = \begin{cases} 1 & \textit{success} \\ 0 & \textit{failure} \end{cases}$$

$$P(X = 1) = p \qquad \text{and} \qquad P(X = 0) = 1 - p$$

where p is the success probability,

9

Binary Outcomes

Bernoulli Trial:
Random experiment with just two outcomes — **success/failure**
heads/tails; yes/no; death/survival; ...

For a single trial, random variable

$$X = \begin{cases} 1 & \textit{success} \\ 0 & \textit{failure} \end{cases}$$

$$P(X = 1) = p \qquad \text{and} \qquad P(X = 0) = 1 - p$$

where p is the success probability, or more compactly

$$P(X = x) = p^x(1 - p)^{1-x} \quad x = 0, 1$$

10

Binary Outcomes

Bernoulli Trial:
Random experiment with just two outcomes — **success/failure**
heads/tails; yes/no; death/survival; ...

For a single trial, random variable

$$X = \begin{cases} 1 & \textit{success} \\ 0 & \textit{failure} \end{cases}$$

$$P(X = 1) = p \qquad \text{and} \qquad P(X = 0) = 1 - p$$

where p is the success probability, or more compactly

$$P(X = x) = p^x(1 - p)^{1-x} \quad x = 0, 1$$

Mean: $E[X] = (0)(1 - p) + (1)p = p$

Variance: $\mathrm{Var}(X) = p(1-p)$

11

Binary Outcomes — Sequence of Bernoulli Trials

- outcomes of trials mutually **independent**
- probability of success p is **constant** over trials

Note independence and constant success probability may not always be appropriate assumptions.

12

Motivating Example: Camera Flash Tests

The time to recharge the flash is tested in **three** mobile phone cameras. The **probability** that a camera passes the test is **0.8**, and the cameras perform independently.

The random variable *X* denotes the number of cameras that pass the test. The last column of the table shows the values of *X* assigned to each outcome of the experiment.

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	*X*
Pass	Pass	Pass		3
Fail	Pass	Pass		2
Pass	Fail	Pass		2
Fail	Fail	Pass		1
Pass	Pass	Fail		2
Fail	Pass	Fail		1
Pass	Fail	Fail		1
Fail	Fail	Fail		0

*What is the probability that the **first and second cameras pass** the test and the third one fails ?*

P(*PPF*) = ?

13

13

Motivating Example: Camera Flash Tests

*What is the probability that the **first and second cameras pass** the test and the third one fails ?*

P(*PPF*) = (0.8)(0.8)(0.2) = 0.128

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	*X*
Pass	Pass	Pass	0.512	3
Fail	Pass	Pass	0.128	2
Pass	Fail	Pass	0.128	2
Fail	Fail	Pass	0.032	1
Pass	Pass	Fail	0.128	2
Fail	Pass	Fail	0.032	1
Pass	Fail	Fail	0.032	1
Fail	Fail	Fail	0.008	0
			1.000	

Each camera test can be treated as a Bernoulli trial.
Probabilities for all other outcomes calculated in a similar fashion.

What is the probability that two cameras pass the test in three trials ?

14

14

Motivating Example: Camera Flash Tests

What is the probability that two cameras pass the test in three trials ?

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	*X*
Pass	Pass	Pass	0.512	3
Fail	Pass	Pass	0.128	2
Pass	Fail	Pass	0.128	2
Fail	Fail	Pass	0.032	1
Pass	Pass	Fail	0.128	2
Fail	Pass	Fail	0.032	1
Pass	Fail	Fail	0.032	1
Fail	Fail	Fail	0.008	0
			1.000	

How many **ways** can this event happen ?

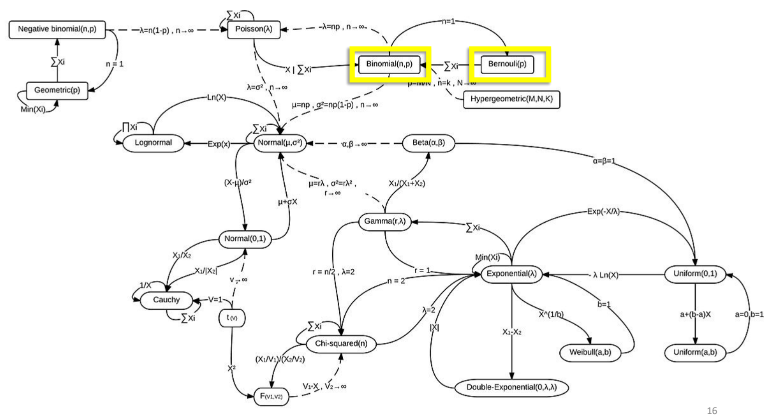
$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{3!}{2!\,(3-2)!} = \frac{3.2.1}{2.1.1} = 3$$

What is the probability of this event ?
0.128 for **each** of the three ways
probability = 3(0.128) = 0.384

This is an example of the Binomial Distribution.

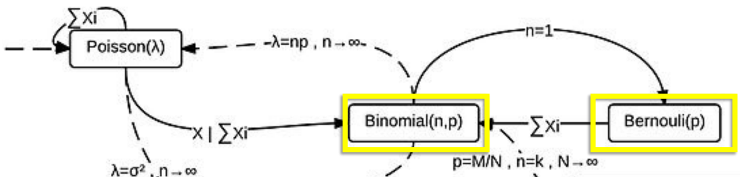
15

15



16

16



17

17

Binomial Distribution

A random experiment consists of n Bernoulli trials such that

(1) The trials are independent

(2) Each trial results in only two possible outcomes, labeled as "success" and "failure"

(3) The probability of a success in each trial, denoted as p, remains constant

The random variable X that equals the number of trials that result in a success has a **binomial random variable** with parameters $0 < p < 1$ and $n = 1, 2, \ldots$. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n \qquad \text{(3-7)}$$

18

18

Motivating Example: Camera Flash Tests

Calculate the probability of 2 passes in three tests.

We are given that n = 3 and *p* = 0.8.

Use the Binomial distribution formula where X is the number of passes:

$$P(X = 2) =$$

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	X
Pass	Pass	Pass	0.512	3
Fail	Pass	Pass	0.128	2
Pass	Fail	Pass	0.128	2
Fail	Fail	Pass	0.032	1
Pass	Pass	Fail	0.128	2
Fail	Pass	Fail	0.032	1
Pass	Fail	Fail	0.032	1
Fail	Fail	Fail	0.008	0
			1.000	

19

19

Motivating Example: Camera Flash Tests

Calculate the probability of 2 passes in three tests.

We are given that n = 3 and *p* = 0.8.

Use the Binomial distribution formula where X is the number of passes:

$$P(X = 2) = \binom{3}{2}(0.8)^2(0.2)^1$$
$$= 3(0.128)$$
$$= 0.384$$

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	X
Pass	Pass	Pass	0.512	3
Fail	Pass	Pass	0.128	2
Pass	Fail	Pass	0.128	2
Fail	Fail	Pass	0.032	1
Pass	Pass	Fail	0.128	2
Fail	Pass	Fail	0.032	1
Pass	Fail	Fail	0.032	1
Fail	Fail	Fail	0.008	0
			1.000	

20

20

Exercise: Organic Pollution

Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant.

Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.

21

Exercise: Organic Pollution

Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.

Let X denote the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with $p = 0.1$ and $n = 18$

$$P(X = 2) =$$

22

Exercise: Organic Pollution

Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.

Let X denote the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with $p = 0.1$ and $n = 18$

$$P(X = 2) = \binom{18}{2}(0.1)^2(0.9)^{16} = 153(0.1)^2(0.9)^{16} = \underline{0.2835}$$

23

Using R to calculate probabilities from a Binomial Distribution: dbinom function

dbinom(x, size, prob)

x is the number of events of interest required,
size is the total number of trials,
prob is the probability of the event occurring.

24

Using R to calculate probabilities from a Binomial **D**istribution: **d**binom function

In the Organic Pollution example x=2, size=18 and p=0.10

```
dbinom(x=2, size=18, prob=0.1)
```

```
0.2835121
```

25

25

Exercise: Organic Pollution revisited

Determine the probability that $3 \le X < 7$.

$X = 3, 4, 5, 6$

$P(3 \le X < 7) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= \binom{18}{3} 0.1^{3} 0.9^{15} + \binom{18}{4} 0.1^{4} 0.9^{14} + \binom{18}{5} 0.1^{5} 0.9^{13} + \binom{18}{6} 0.1^{6} 0.9^{12}$

26

26

Exercise: Organic Pollution revisited

Now determine the probability that $3 \le X < 7$.

Answer:

$$P(3 \le X < 7) = \sum_{x=3}^{6} \binom{18}{x} (0.1)^{x} (0.9)^{18-x}$$
$$= 0.168 + 0.070 + 0.022 + 0.005$$
$$= 0.265$$

27

27

```
sum(dbinom(x=3:6, size = 18, prob=0.1))
```

```
0.2650319
```

28

28

Binomial Mean and Variance

If X is a binomial random variable with parameters p and n,

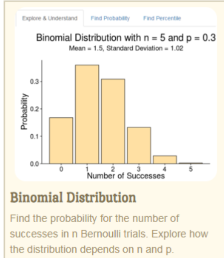
The mean and variance of the binomial distribution $b(x; n, p)$ are
$$\mu = np \text{ and } \sigma^2 = npq.$$

Where q = 1-p.

29

29

Distributions: Explore the Shape, Find Probabilities and Percentiles

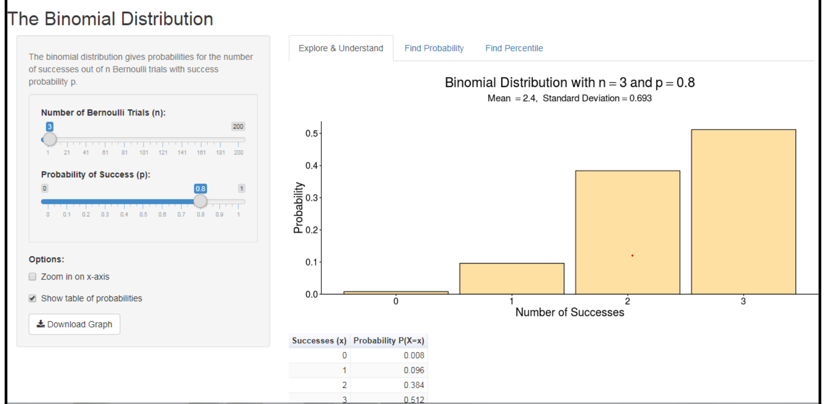


http://www.artofstat.com

Use this app to explore different scenarios for a random variable following a Binomial distribution

30

30

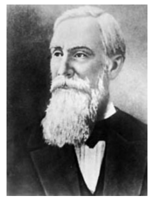


31

31

Chebyshev's Inequality

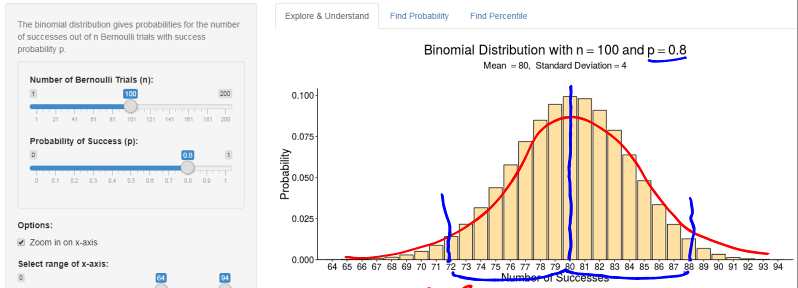
- Chebyshev's inequality provides an estimate as to where a certain % of observations will lie relative to the mean once the **standard deviation** is known.
- For example, at ***least*** 75% of values will lie within two standard deviations of the mean.



32

32

The Binomial Distribution



$E(X) = 100 \times 0.8 = 80$

$Var(X) = 100 \times 0.8 \times 0.2 = 16 \Rightarrow SD(X) = 4$

33

33

StatsConsulting.com

- A medical device company needed to calculate the probability that a particular component of their device fails. They have limited bench data which suggests that the probability of failure is 0.15.
- The plan to test 10 devices and want an indication as to the proportion of failures they should expect to see across all devices in the trial.
- What is the number of failures they can *expect* in 10 devices given the probability of failure of a particular device ?

34

34

Binomial Mean and Variance

If X is a binomial random variable with parameters p and n,

The mean and variance of the binomial distribution $b(x; n, p)$ are
$$\mu = np \text{ and } \sigma^2 = npq.$$

where q=1-p.

35

35

StatsConsulting.com

- The random variable ***X* denotes the number of devices that fail.**
- n = 10 trials and p = 0.15

Use the Binomial distribution.

The typical value they can expect is the mean of the random variable X in question.

36

36

StatsConsulting.com

- The random variable ***X* denotes the number of devices that fail.**
- $n = 10$ trials and $p = 0.15$

Use the Binomial distribution.

The typical value they can expect is the mean of the random variable X in question.

$\mu = np = 10*0.15 = 1.5$

i.e. they can expect 1.5 devices to fail in a sample of 10 …. interpret this !

37

37

StatsConsulting.com

- The random variable ***X* denotes the number of devices that fail.**
- $n = 10$ trials and $p = 0.15$

Use the Binomial distribution.

The variance they can expect is

$\sigma^2 = np(1-p) = 10*0.15*(1-0.15) = 1.27$

The standard deviation is the square root of 1.27 = 1.13

Use Chebyshev's inequality to interpret this !

38

38

Motivating Example: Camera Flash Tests

- The random variable ***X* denotes the number of cameras that pass the test**.
- $n = 3$ and $p = 0.8$

Find the mean and variance of the binomial random variable.

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	*X*
Pass	Pass	Pass	0.512	3
Fail	Pass	Pass	0.128	2
Pass	Fail	Pass	0.128	2
Fail	Fail	Pass	0.032	1
Pass	Pass	Fail	0.128	2
Fail	Pass	Fail	0.032	1
Pass	Fail	Fail	0.032	1
Fail	Fail	Fail	0.008	0
			1.000	

39

39

Motivating Example: Camera Flash Tests

- The random variable ***X* denotes the number of cameras that pass the test**.
- $n = 3$ and $p = 0.8$

Find the mean and variance of the binomial random variable.

Camera Flash Tests				
Outcome				
Camera #				
1	2	3	Probability	*X*
Pass	Pass	Pass	0.512	3
Fail	Pass	Pass	0.128	2
Pass	Fail	Pass	0.128	2
Fail	Fail	Pass	0.032	1
Pass	Pass	Fail	0.128	2
Fail	Pass	Fail	0.032	1
Pass	Fail	Fail	0.032	1
Fail	Fail	Fail	0.008	0
			1.000	

$\mu = np = 3*0.8 = 2.4$

$\sigma^2 = np(1-p) = 3*0.8*0.2 = 0.48$

$\sigma = SD(X) = 0.69$

40

40

Is the Binomial distribution applicable here ?

Can each trial can be summarized as resulting in either a success or a failure with a fixed probability, assumed independent from trial to trial ?

- A multiple choice test contains 10 questions, each with four choices, and you guess at each question. Let X= the number of questions answered correctly.
- In the next 20 births at a hospital, let X= the number of female births.
- A worn machine tool produces 1% defective parts. Let X=number of defective parts in the next 25 parts produced.
- The probability of ordering a hot chocolate in Mr Waffle is 0.10. A group enters a coffee shop and each member places an order. Let X=number of **hot chocolates** ordered.

41

41

Summary so far

- Bernoulli trials and Binomial distribution
- dbinom (in R) and sum(dbinom(start:fininsh, size=, p=) trick
- Mean = np, var=np(1-p)
- When the binomial does and does not apply.
- Oliver's world.

42

42

Oliver's world

10,000 products made daily
Probability of a complaint is 0.0001.

What is the probability Oliver will see 10 complaints in a day ?

Does the Binomial Distribution apply ?

If you assume it does …. Let X be a random variable representing the number of complaints Oliver will receive in a day.
You are given that n = 10,000 and p=0.0001

43

43

Oliver's world

P(X=10)= $\binom{n}{x} p^x (1-p)^{n-x} = \binom{10,000}{10} 0.0001^{10} (1-0.0001)^{10,000-10}$

$$= 0.0000001010183$$

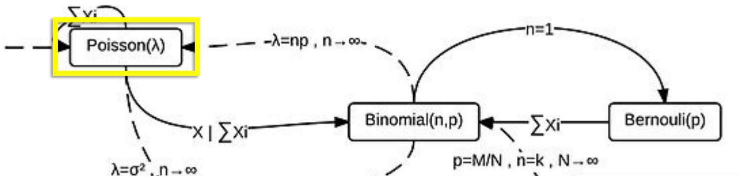
```
dbinom(x=10, size=10000, prob= 1/10000)
```

```
dbinom(x=10, size=10000, prob= 1/10000)
 1.010183e-07
```

44

44

Poisson Distribution



45

45

Poisson Distribution

- Experiments yielding numerical values of a random variable X, the number of outcomes occurring during a given time interval or in a specified region, are called Poisson experiments.
- The given time interval may be of any length, such as a minute, a day, a week, a month, or even a year.
- A Poisson experiment is derived from the Poisson process and possesses the following properties.

46

46

Properties of the Poisson Process

- The number of **outcomes** occurring in one time interval or specified region of space is **independent** of the number that occur in any other disjoint time interval or region. In this sense we say that the Poisson process has no memory.
- The **probability** that a **single outcome** will occur during a very short time interval or in a small region is **proportional** to the **length** of the time interval or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.
- The probability that **more than one outcome** will occur in such a short time interval or fall in such a small region is **negligible**.

47

47

Poisson Distribution

The random variable *X* that equals the **number of events** in a Poisson process is a Poisson random variable with parameter λ > 0, and the probability density function is:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for} \quad x = 0, 1, 2, 3, \ldots$$

48

48

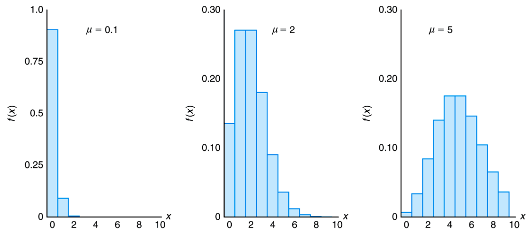
Mean and Variance of Poisson Distribution

- If λ is the average number of successes occurring in a given time interval or region in the Poisson distribution, then the mean and the variance of the Poisson distribution are both equal to λ.
- Mean = λ, variance = λ
- A one parameter distribution.

49

49

Poisson density functions for different means

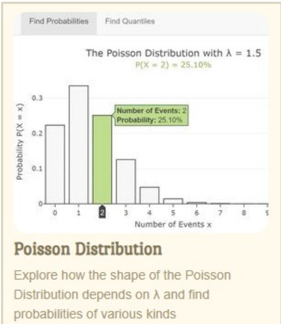


If the variance is much greater than the mean, then the Poisson distribution would not be a good model for the distribution of the random variable.

50

50

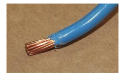
Distributions: Explore the Shape, Find Probabilities and Percentiles



51

51

Poisson Example: Calculations for Wire Flaws



Suppose that the number of flaws on a thin copper wire follows a Poisson distribution with a mean of 2.3 flaws per mm.

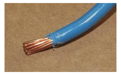
Find the probability of exactly 2 flaws in 1 mm of wire.

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for} \quad x = 0,1,2,3,...$$

52

52

Poisson Example: Calculations for Wire Flaws



Suppose that the number of flaws on a thin copper wire follows a Poisson distribution with a mean of 2.3 flaws per mm.

Find the probability of exactly 2 flaws in 1 mm of wire.

$$P(X=2)=\frac{e^{-2.3}2.3^{2}}{2!}=0.265$$

53

53

Using R to calculate probabilities from a Poisson Distribution: dpois

dpois(x, lambda)

x is the number of events of interest,
lambda is the mean.

54

54

Using R to calculate probabilities from a Poisson Distribution: dpois

dpois(x, lambda)
x is the number of events of interest, lambda is the mean

Copper wire example: x=2, lambda= 2.3 flaws per mm
The probability of exactly 2 flaws in 1 mm of wire

```
dpois(x=2, lambda =2.3)
0.2651846
```

55

55

Example: Calculations for Wire Flaws revisited

Suppose that the number of flaws on a thin copper wire follows a Poisson distribution with a mean of 2.3 flaws per mm.

Determine the probability of 10 flaws in **5 mm** of wire.

56

56

Determine the probability of 10 flaws in **5** mm of wire.

Let X denote the number of flaws in 5 mm of wire. We know that there will be 2.3 per 1mm therefore we expect 2.3 X 5 = 11.5 flaws per 5 mm.

$$P(X = 10) = e^{-11.5} \frac{11.5^{10}}{10!} = 0.113$$

```
dpois(x=10, lambda =2.3*5)
0.1129351
```

57

57

Example: Car Park

A car park has 3 entrances, A, B and C.

The number of cars per hour entering through each of these is Poisson distributed with means $\lambda_A = 1.5$, $\lambda_B = 1.0$, $\lambda_C = 2.5$.

Arrivals at each entrance are **independent**.

58

58

Example: Car Park

A car park has 3 entrances, A, B and C.

The number of cars per hour entering through each of these is Poisson distributed with means $\lambda_A = 1.5$, $\lambda_B = 1.0$, $\lambda_C = 2.5$.

Arrivals at each entrance are **independent**.

T = Total number of cars entering in an hour

59

59

Example: Car Park

A car park has 3 entrances, A, B and C.

The number of cars per hour entering through each of these is Poisson distributed with means $\lambda_A = 1.5$, $\lambda_B = 1.0$, $\lambda_C = 2.5$.

Arrivals at each entrance are **independent**.

T = Total number of cars entering in an hour

$$T \sim \text{Poisson}(\lambda_A + \lambda_B + \lambda_C) \equiv \text{Poisson}(1.5 + 1.0 + 2.5) \equiv \text{Poisson}(5)$$

$$P(T = 4) = \frac{e^{-5} 5^4}{4!} = 0.1755$$

60

60

Sums of Independent Poisson Random Variables

If $X_1, X_2 \ldots, X_n$ are independently Poisson distributed with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ then

$$T = X_1 + X_2 + \cdots + X_n \quad \text{is} \quad \text{Poisson}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

and

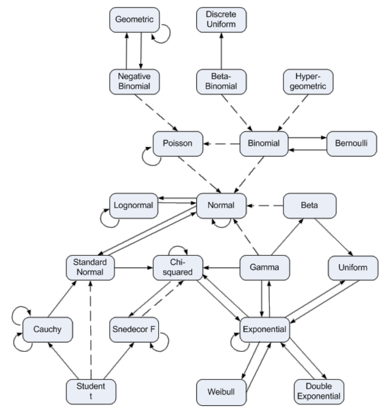
$$E[T] = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

and

$$\text{Var}(T) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

61

61



https://www.johndcook.com/blog/distribution_chart/

62

The big three ….

- Binomial Distribution
  - In a study involving testing the effectiveness of a new drug, the number of cured patients among all the patients who use the drug approximately follows a binomial distribution
- Geometric Distribution
  - In a statistical quality control problem, the experimenter will signal a shift of the process mean when observational data exceed certain limits. The number of samples required to produce a false alarm follows a geometric distribution.
- Poisson Distribution
  - The number of white cells from a fixed amount of an individual’s blood sample is usually random and may be described by a Poisson distribution.

63

63