Topic 4: Probability

First, an introduction: Dr Nicola Fitz-Simon

- Studied statistics at TCD, PhD (2006)
- Worked as a statistician on research studies and lecturing in the UK at Oxford University, LSHTM, Imperial College London
- Most recent post in Galway in the Clinical Research Facility
- Research interests in statistical methods for causal inference
- Contact details next week ...

Summary so far

- Statistical inference involves sampling from populations to generate an estimate (i.e. a statistic) of a population parameter of interest.
- Choosing a sample at *random* is crucial. Subjective sampling will lead to bias (e.g. circles example).

The challenge ...

- It is well and good to see what happens if you take lots of samples at random from a population.
- In practice you will only be taking one sample !
- What can you say about how likely your statistic is to be a good guess of the population parameter of interest ?
- To answer this we need to look at some probability theory.

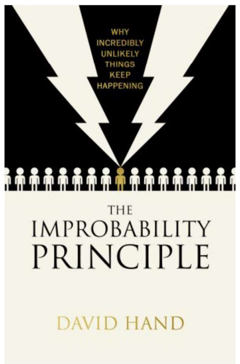
The Role of Probability

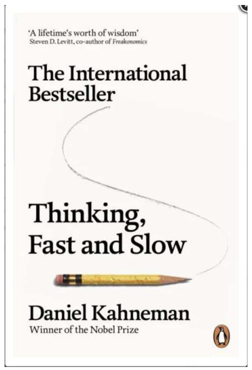
- Probability provides the **framework** for the study and application of statistics.

- Probability concepts will be introduced in the next few lectures.

Learning Objectives

1. Interpret probabilities and calculate probabilities of events
2. Calculate the probabilities of joint events
3. Interpret and calculate conditional probabilities
4. Determine independence and use independence to calculate probabilities
5. Understand Bayes' theorem and when to use it





Probability:

How likely .... ?

Tossing a coin

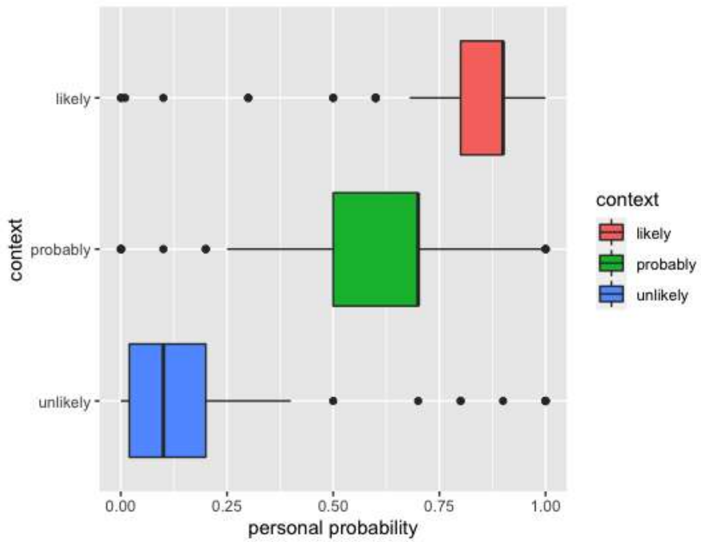
- If I toss a coin, what is the probability it will turn up heads?

Tossing a coin

Magician and statistician Persi Diaconis found that when tossing a coin and catching it in the hand the probability of the same face turning up as initially is about
0.51. https://www.youtube.com/watch?v=AYnJv68T3MM

What are probabilities?

- 6-sided die about to be tossed for the first time
- **Classical**: 6 possible outcomes, by symmetry each equally likely to occur
- **Frequentist**: Empirical evidence shows that similar dice thrown in the past have landed on each side about equally often
- **Subjective**: the degree of individual belief in occurrence of an event – can be influenced by classical or frequentist arguments, eg here may be willing to bet at a rate of 1/6 on any side
- Subjective probabilities also influenced by other reasons when symmetry arguments don't apply and repeated trials are not possible



Probability

- The probability of an event **A** is the number of (equally likely and disjoint) outcomes in the event divided by the total number of (equally likely and disjoint) possible outcomes.

$$P(\mathbf{A}) = \frac{\text{\# of outcomes in } \mathbf{A}}{\text{\# of possible outcomes}}$$

$(0 \leq P(A) \leq 1)$ **

All possible outcomes (Sample spaces)

- The set of all possible outcomes of a random experiment is called the sample space, *S*.
- *S* is discrete if it consists of a finite or countable infinite set of outcomes.
- *S* is continuous if it contains an interval of real numbers.
- P(S)=1 **

Examples of Sample Spaces

- Toss a coin twice
  S = { HH, HT, TH, TT }
- Roll a pair of dice and record numbers
  S = {(1,1),(1,2),…,(1,6),(2,1),…, (2,6),…,(6,6)}
- Roll a pair of dice and record total score
  S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
- Toss a coin until first tail appears
  S = {T, HT, HHT, HHHT, …}
- Measure duration of charge of mobile phone battery
  $S = \{ t \mid t \geq 0 \}$

Events

An **Event** is a specific collection of sample points / possible outcomes.

An event is denoted by **E** or capital letters at the start of the alphabet, A, B, C etc.

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A **Simple Event** is a collection of only one sample point/possible outcome

- Eg: Throw a die – Event 1 : get a 4

  E_1={4} - a simple event

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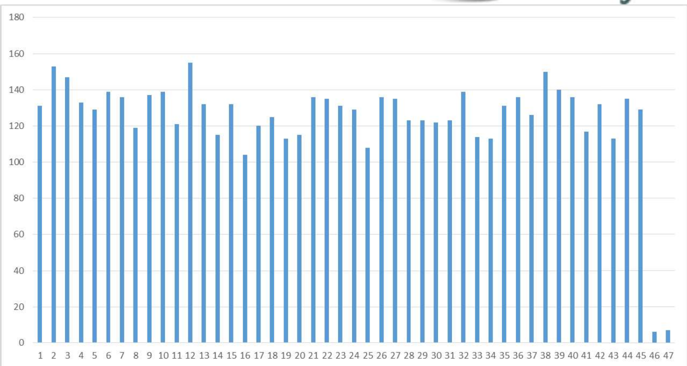
A **Compound Event** is a collection of more than one sample point/possible outcomes

- Eg: Throw a die – Event 2: get at least a 4

  E_2={4,5,6} - a compound event

Example: Lottery



Since launch of Lotto 6/45 on November 4th 2006.
Balls 46 and 47 introduced September 3rd 2015.

Learning to count ..

- How many outcomes are there for the lotto when the outcome of interest is to guess 6 numbers correctly from 47 ?
- How many ways can this occur ?
- Combinatorics: permutations and combinations

To calculate a probability:

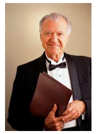
If each sample point in the sample space is equally likely

$$P(\text{Event}) = \frac{\textit{Number of outcomes in event}}{\textit{Total number outcomes in sample space.}}$$

So we need to learn how to count….

Counting by Multiplication

The fixed-price dinner at a restaurant provides the following choices:



Appetizer: Soup or Salad
Main Course: Baked chicken,
Broiled beef patty,
Baby beef liver,
or Roast beef
Dessert: Ice-cream or Cheese cake

How many different three course meals can be ordered?

Counting by Multiplication

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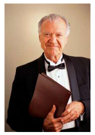

Appetizer: Soup or Salad
Main Course: Baked chicken,
Broiled beef patty,
Baby beef liver,
or Roast beef
Dessert: Ice-cream or Cheese cake

How many different three course meals can be ordered? 2x4x2=16

The multiplication principle

If a task consists of a sequence of choices in which there are
p selections for the first choice,
q selections for the second choice,
r selections for the third choice,
and so on,
then the task of making these selections can be done in

$p \times q \times r$ ..

different ways.

Example: Postal delivery

You have just been hired as a Post Delivery person for University of Galway. On your first day, you must travel to seven buildings with letters.

How many different routes are possible?

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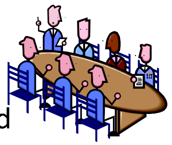
How many different routes are possible?

7x6x5x4x3x2x1=5040

Example: Committee Problem

Three members from a 14-member committee are to be randomly selected to serve as chair, vice chair, and secretary.



The first person selected is the chair, the second person selected is to be vice chair, and the third secretary.

How many different committee structures are possible?

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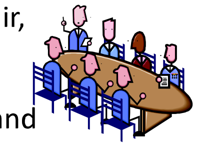


The first person selected is the chair, the second person selected is to be vice chair, and the third secretary.

How many different committee structures are possible? 14x13x12=2148

Permutations

A **permutation** is an *arrangement* of objects.

We have seen that arranging *n* distinct (different) objects can be done in $n(n-1)(n-2)\ldots3.2.1$ different ways.

This calculation is often written using the ***factorial*** symbol. If *n* is an integer, the factorial symbol $n!$ is defined as $n! = n(n-1)(n-2)\ldots3.2.1$

E.g. $3! = 3.2.1 = 6$ E.g. $2! = 2.1 = 2$

Example: Postal delivery

You have just been hired as a Post Delivery person for University of Galway. On your first day, you must travel to seven buildings with letters.

How many different routes are possible?

7x6x5x4x3x2x1=5040=7!

```
>
> factorial(7)
[1] 5040
```


Permutations

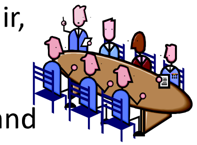
A permutation can also be an arrangement of *r* objects chosen from *n* distinct (different) objects where replacement in the selection is not allowed.

The symbol, P_r^n, represents the number of permutations of *r* objects selected from *n* objects.

The calculation is given by the formula:

$$P_r^n = \frac{n!}{(n-r)!}$$

Example: Committee Problem



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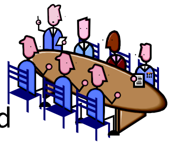


$$P_3^{14} = \frac{14!}{(14-3)!}$$

```
>
> factorial(14)/factorial(14-3)
[1] 2184
>
```


Example: Renmore U12 Soccer

- 6 players available, {A,B,C,D,E,F}
- 5 a side competition
- Number of permutations of 6 players choosing 5 at a time ?

Renmore U12 Soccer

- 6 players available, {A,B,C,D,E,F}
- 5 a side competition
- Number of permutations of 6 players choosing 5 at a time ?

$${}^6P_5 = \frac{n!}{(n-r)!} = \frac{6!}{(6-5)!} = \frac{6!}{1!} = 720$$

```
> factorial(6)/factorial(6-5)
[1] 720
```


Renmore U12 Soccer

- 6 players available, {A,B,C,D,E,F}
- 5 a side competition
- 720 permutations – be careful as order doesn't matter here !

Team A,B,C,D,E is the same team as E,D,C,B,A ….. lots of double counting

Combinations (when order doesn't matter!)

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Renmore U12 Soccer

- 6 players available, {A,B,C,D,E,F}
- 5 a side competition
- Number of combinations (as order doesn't matter) ?

Renmore U12 Soccer

- 6 players available, {A,B,C,D,E,F}
- 5 a side competition
- Number of combinations ?

$$^6C_5 = \frac{n!}{r!(n-r)!} = \frac{6!}{5!(6-5)!} = \frac{6!}{5!1!} = 6$$

```
choose(6,5)
[1] 6
```

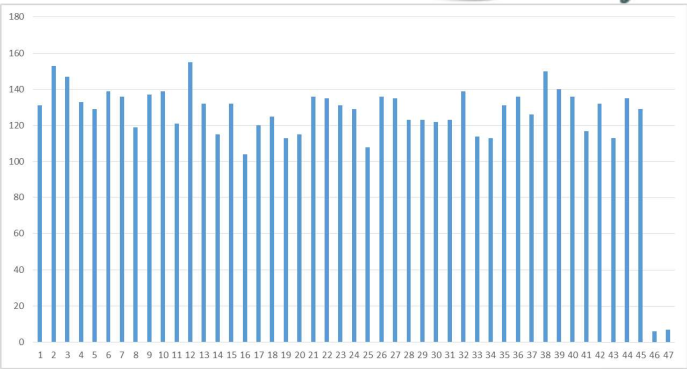

Renmore U12 Soccer

- 6 players available, {A,B,C,D,E,F}
- 5 a side competition
- 6 teams to choose

{A,B,C,D,E}, {A,B,C,D,F}, {A,B,C,E,F}, {A,B,D,E,F}, {A,C,D,E,F}, {B,C,D,E,F}

- Total football, every child gets a chance … what is the probability of any team of this list being the one chosen to start ?

Example: Lottery



Since launch of Lotto 6/45 on November 4th 2006.
Balls 46 and 47 introduced September 3rd 2015.

Knowing when order matters is important …

- https://www.youtube.com/watch?v=wOLxoCF19Ng

Winning the Lotto

- 47 balls available, {1,2,3, … ,47}
- 6 are selected at random
- Number of combinations ?

$$^{47}C_6 = \frac{n!}{r!\,(n-r)!} =$$

Winning the Lotto

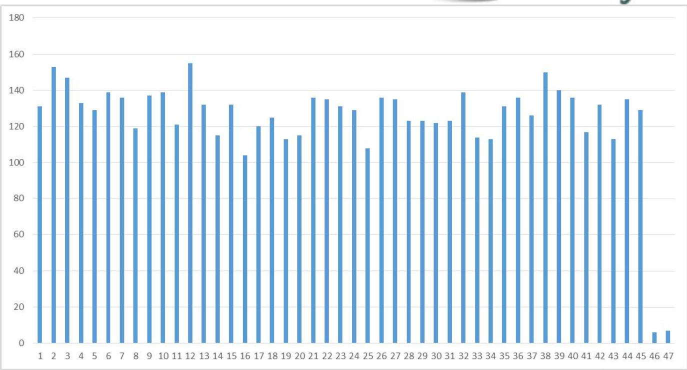
- 47 balls available, {1,2,3, … ,47}
- 6 are selected at random
- Number of combinations ?

$$^{47}C_6 = \frac{n!}{r!(n-r)!} = \frac{47!}{6!(47-6)!} = \frac{47!}{6!41!}$$

= 10737573

```
> choose(47,6)
[1] 10737573
```

National Lottery



Since launch of Lotto 6/45 on November 4th 2006.
Balls 46 and 47 introduced September 3rd 2015.

```
> choose(47,6)
[1] 10737573
```


National Lottery

For a 4 euro bet, a player fills two lines - two sets of 6 numbers.

What is the probability of winning the Irish Lotto with a 4 euro bet if ordering is not important?

National Lottery

For a 4 euro bet, a player fills two lines - two sets of 6 numbers.

What is the probability of winning the Irish Lotto with a 4 euro bet if ordering is not important?

With two lines there are two chances

$$P(win) = \frac{2}{10737573} = 0.0000001862618$$

Repetition: *n* non-distinct elements

The number of permutations of n of which
n_1 are of one kind,
n_2 are of a second kind,
...,
and n_k are of a kth kind
is given by

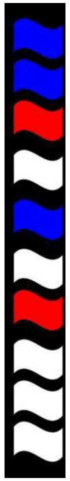
$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

where $n_1 + n_2 + ... + n_k = n$.

Repetition example: flags

How many different vertical arrangements are there of 10 flags if 5 are white, 3 are blue and 2 are red?

Solution:



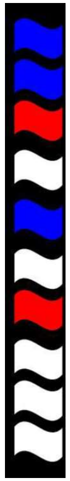
Example: flags

How many different vertical arrangements are there of 10 flags if 5 are white, 3 are blue and 2 are red?

Solution:

$$\frac{10!}{5!\,3!\,2!} = 2520$$

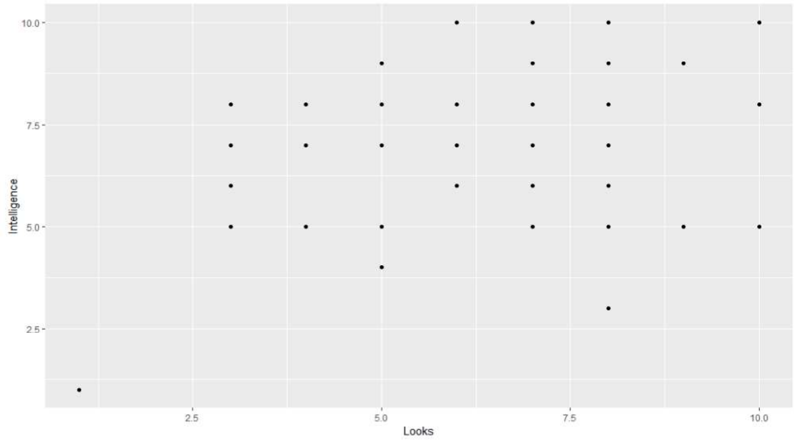
```
> factorial(10)/(factorial(5)*factorial(3)*factorial(2))
[1] 2520
>
```



Joint Events ....

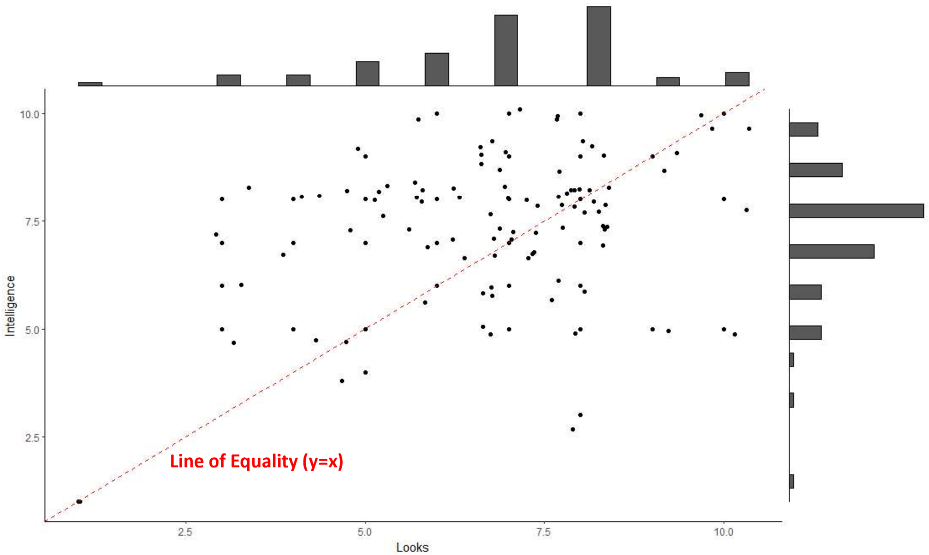
- Class survey …

Simple scatterplot

```
```{r cars}
ST2001.Data.Sc %>% ggplot(aes(y=Intelligence, x=Looks))+
  geom_point()
```
```

Simple scatterplot



A better scatterplot

```
```{r cars}
p <- ST2001.Data.Sc %>% ggplot(aes(y=Intelligence, x=Looks))+
        geom_point()+
        geom_jitter()+
        geom_abline(intercept=0, slope=1, linetype="dashed", color= "red")+
        theme_classic()
p <- ggExtra::ggMarginal(p, type="histogram")
p

```
```

Points are jittered (i.e. no longer hidden behind each other) and a line of equality (i.e. y=x) is added as a reference and (marginal) histograms for each variable displayed.



Joint Events ….

- Write down 9 characteristics that your ideal person in life must have.
- Assign a probability to each event.
- Work out the probability of meeting a person with ***all*** characteristics you have listed.

Joint events (and / or)

- Probabilities of <u>joint events</u> can often be determined from the probabilities of the individual events that comprise them.

- Joint events are generated by applying basic set operations to individual events, specifically:
  - Complement of event A is
    $\bar{A}$ = all outcomes ***not*** in A
  - A∪B – ***Union*** of events; A or B or both
  - A∩B – ***Intersection*** of events A and B

  - ***Disjoint*** events cannot occur together, i.e. A∩B = ∅

Example: Rolling a die

- A = score on die is even = { }
- B = score on die is odd = { }
- C = score is greater than 4 = { }
- A∩B =
- A∪B =
- A∩C =
- B∩C =
- (A∩C) ∪ (B∩C) =
- (A∪B) ∩C =

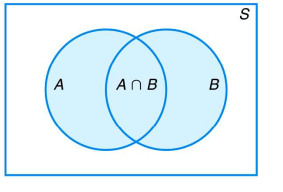
Example: Rolling a die

- A = score on die is even = {2, 4, 6 }
- B = score on die is odd = {1, 3, 5 }
- C = score is greater than 4 = { 5, 6 }
- A∩B = {}= ∅
- A∪B = {1, 2, 3, 4, 5, 6}=S
- A∩C = {6}
- B∩C = {5}
- (A∩C) ∪ (B∩C) = {5,6}
- (A∪B) ∩C = S∩C=C={5,6}

Probability of a Union

- For **any two events** *A* and *B*, the probability of union is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example ?

Probability of a Union: disjoint events

- For two **disjoint** events **A** and **B**, the probability that one *or* the other occurs is the sum of the probabilities of the two events.

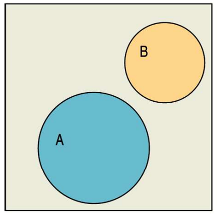
$$P(A \cup B) = P(A) + P(B)$$

provided that **A** and **B** are disjoint.

(also called mutually exclusive)

Example ?

Disjoint event share no common outcomes
$A \cap B = \emptyset$



Two disjoint sets, **A** and **B.**

Example …. exam paper 2018

b) A Garda report claims that 78% of drivers who are stopped on suspicion of drunk driving are given a breath test, 36% a blood test and 22% both tests. What is the probability that a randomly selected suspected driver is given a test?

Example …. exam paper 2018

b) A Garda report claims that 78% of drivers who are stopped on suspicion of drunk driving are given a breath test, 36% a blood test and 22% both tests. What is the probability that a randomly selected suspected driver is given a test?

b) $0.78 + 0.36 - 0.22 = 0.92$

Example: Draw a card; event A – an Ace; event B – a heart

	B	$\bar{B}$	Total
A	$\frac{1}{52}$	$\frac{3}{52}$	$\frac{4}{52}$
$\bar{A}$	$\frac{12}{52}$	$\frac{36}{52}$	$\frac{48}{52}$
Total	$\frac{13}{52}$	$\frac{39}{52}$	1.00

Table of joint probabilities

	B	$\bar{B}$	Total
A	$P(A \cap B)$	$P(A \cap \bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A} \cap B)$	$P(\bar{A} \cap \bar{B})$	$P(\bar{A})$
Total	$P(B)$	$P(\bar{B})$	1.00

Are events disjoint (mutually exclusive) ?

- If P(A U B) is greater than 1 then you know you have made a mistake and the events were not mutually exclusive (i.e. there is an intersection).
- Domain knowledge is needed here …

Intersections (A and B)

Multiplication Rule for independent events:

For two independent events **A** and **B**, the probability that *both* **A** and **B** occur is the product of the probabilities of the two events

$$P(A \cap B) = P(A) \times P(B)$$

provided that **A** and **B** are independent.

This means that occurrence of one event has no impact on the probability of occurrence of the other event.

Example: Electronic components

Two electronic components are selected at random from a production line for inspection. It is known that 90% of the components have no defects.

What is the probability that the two inspected components have no defects?

Calculate probability of intersection of A and B

Let A = 1^{st} component no defect, B=2^{nd} component has no defect

$$P(A \cap B)\ ?$$

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ &= 0.90 \cdot 0.90 \\ &= 0.81 \end{aligned}$$

Example: Electronic components with dependence

What if the probability of the second component having no defects changes once we know that the first component had no defects ?

How might this arise ?

Conditional Probability

- $P(B \mid A)$ is the probability of event B occurring, given that event A has already occurred.

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

Independence revisited

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Example: Electronic components (dependent events)

Two electronic components are selected at random from a production line for inspection. It is known that the probability that the first component has no defects is 0.90 and that the probability that a second component has no defects **given** that the first component had no defects is 0.95.

What is the probability that the two inspected components have no defects?

Calculate probability of intersection A and B

Let A = 1st component no defect, B=2nd component has no defect **given** that A had no defect.

$$P(A \cap B)\ ?$$

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B \mid A) \\ &= 0.90 \times 0.95 \\ &= 0.855 \end{aligned}$$

Knowing that the first was defect free has increased the probability of both being defect free (i.e. from 0.81 to 0.855)

Are events independent ?

- Assuming independence is a HUGE assumption
- The product will be less than 1 so your answer will always make sense but is unlikely to be correct!!

Sally Clark, mother wrongly convicted of killing her sons, found dead at home

- Family says she never recovered from court case
- Cause of death to be determined by coroner

Solicitor Sally Clark and her husband Stephen outside the High Court in central London in 2003. Photograph: Chris Young/PA

Sally Clark, the solicitor wrongly convicted of murdering her two baby sons, was found dead by her family at her home yesterday.

Mrs Clark, 42, who served three years of a life sentence after being found guilty

Professor Sir Roy Meadow, the controversial paediatrician, an expert witness at the trial, told the jury the chance of two children in an affluent family suffering cot death was "one in 73m". The Royal Statistical Society disagreed and wrote to the lord chancellor saying there was "no statistical

Conditional probability when B depends on A

- To find the probability of the event **B** *given* the event **A**, we restrict our attention to the outcomes in **A**. We then find the fraction of *those* outcomes **B** that also occurred.

$$P(\mathbf{B}|\mathbf{A}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{A})}$$

- Note: $P(\mathbf{A})$ cannot equal 0, since we know that **A** has occurred.

Example: musical children

At a parents evening at the local Boys school a parent was overheard to say:

"Both of my children are musical"

What is the probability that this parent has two boys?

General Multiplication Rule with dependent events

- The conditional probability can be rewritten to **further** generalise the **multiplication** rule.

$$P(\mathbf{B}|\mathbf{A}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{A})}$$

Task:

1. **rewrite this formula in terms of $P(A \cap B)$**
2. **Use the fact that $P(A \cap B) = P(B \cap A)$ and see if you can reverse the conditioning …**

General Multiplication Rule

- The conditional probability can be rewritten to **further** generalise the **multiplication** rule.

$$P(A \cap B) = P(A) \cdot P(B|A)$$
$$P(B \cap A) = P(B) \cdot P(A|B)$$

As $P(A \cap B) = P(B \cap A)$ implies

$$P(A).\ P(B|A) = P(B) \cdot P(A|B)$$

Reversing the Conditioning

- This results means that $P(\mathbf{A}|\mathbf{B})$ can be calculated once we know $P(\mathbf{A})$, $P(\mathbf{B}$, and $P(\mathbf{B}|\mathbf{A})$.

- From this information, we can find $P(\mathbf{A}|\mathbf{B})$:

$$P(A|B) = \frac{P(B|A) \bullet P(A)}{P(B)} \text{ for } P(B) > 0$$

Bayes' Theorem

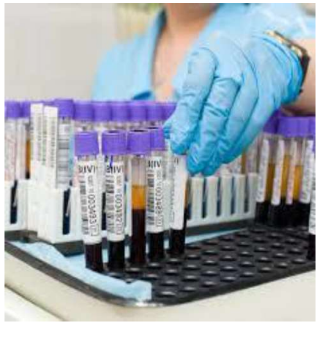
- Thomas Bayes (1702-1761) was an English mathematician and Presbyterian minister.

- Bayes' theorem states that,

$$P(A|B) = \frac{P(B|A) \bullet P(A)}{P(B)} \text{ for } P(B) > 0$$

Recall $P(B \cap A) = P(A \cap B)$ implies $P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$

Diagnostic tests/ Screening



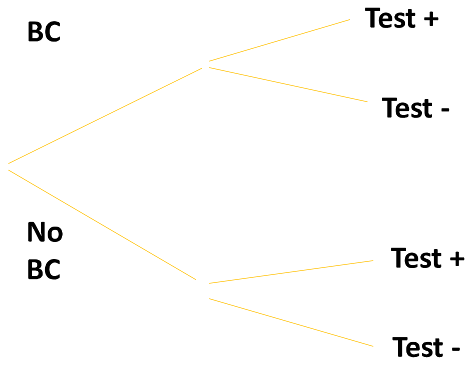
Example: Breast Cancer Screening

- Breast cancer occurs most commonly amongst older women (>60) where it is estimated that 3.65% get breast cancer.
- A mammogram can typically identify correctly 85% of cancer cases (sensitivity) and 95% of cases without cancer (**specificity**).
- If a woman in her 60s gets a positive test what is the probability she has breast cancer?

Breast Cancer Screening tree diagrams

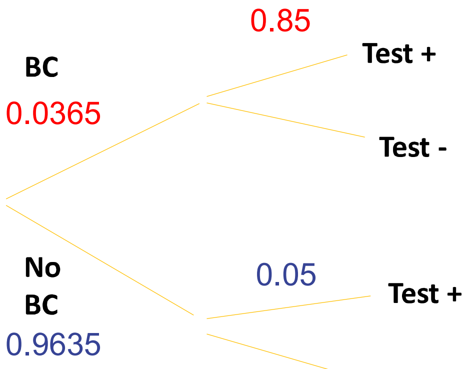
- Think about how many ways can a test come back positive ?
- Tree diagrams are very useful here.

$P(BC \mid +) = ?$



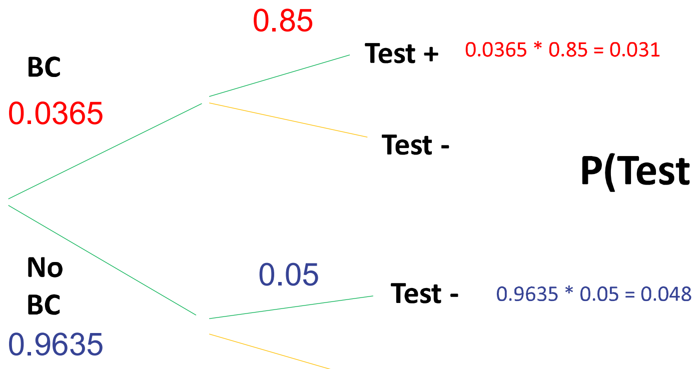
P(Test +) ?

$P(BC \mid +) = ?$



P(Test +) ?

$P(BC \mid +) = ?$



P(Test +) ?

$$P(\mathrm{A} \mid \mathrm{B}) = \frac{P(\mathrm{B}|\mathrm{A}).P(\mathrm{A})\cdot}{P(\mathrm{B})} \quad \text{for } P(\mathrm{B}) > 0$$

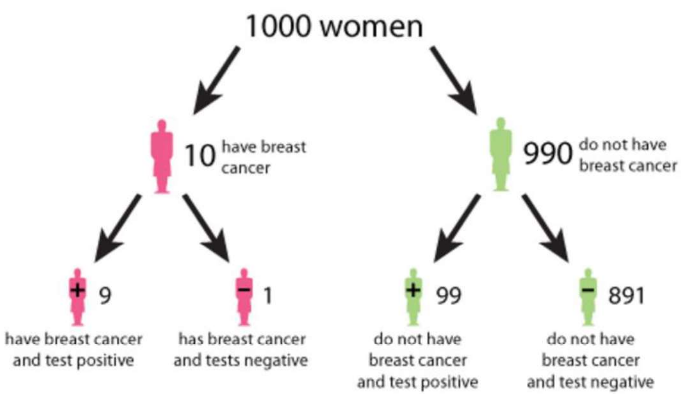
$$P(\mathrm{BC} \mid \text{test} +) = \frac{P(\text{test} + \mid \mathrm{BC}).P(\mathrm{BC})\cdot}{P(\text{test} +)}$$

$$P(\mathrm{BC} \mid \text{test} +) = \frac{(0.85 \times 0.0365)}{(0.85 \times 0.0365) + (0.05 \times 0.9635)}$$

$$P(BC \mid test+) = 0.392$$

Interpretation ?

- The probability of having breast cancer given that the test comes back positive is 0.392.
- How would you communicate this result ?



Example: printer failures

- A printer manufacturer obtained the following three types of printer failure probabilities:

  Hardware P(H) = 0.1,

  Software P(S) = 0.6,

  Other P(O) = 0.3.

Also, previous experiments suggest

P(F | H) = 0.9,

P(F | S) = 0.2,

P(F | O) = 0.5.

If a failure occurs, determine if it's most likely due to hardware, software, or other.

Types of printer failure

If a failure occurs, determine if it's most likely due to hardware, software, or other.

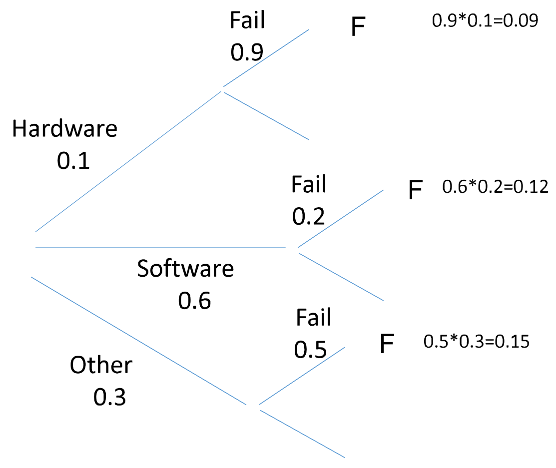
We have P(F | H), P(F | S) and P(F | O)

we need to calculate

P(H | F), P(S | F) and P(O | F)

Start by calculating P(F).

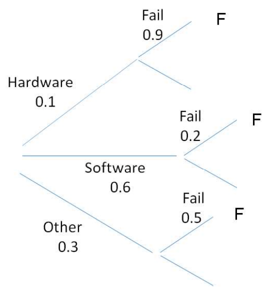
As a tree:



P(F) ?

Calculate probability of failure P(F)

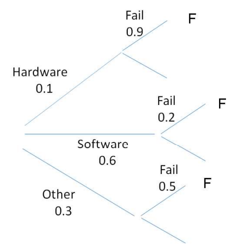
$$P(F) = P(F \mid H)P(H) + P(F \mid S)P(S) + P(F \mid O)P(O)$$
$$= 0.9(0.1) + 0.2(0.6) + 0.5(0.3) = 0.36$$



Now calculate P(H | F) using Bayes Rule

Calculate P(H|F) using Bayes rule

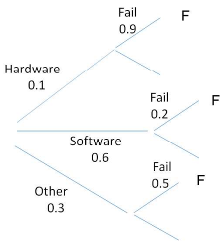
$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.9 \cdot 0.1}{0.36} = 0.250$$



Calculate P(S|F) using Bayes rule

$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.9 \cdot 0.1}{0.36} = 0.250$$

$$P(S \mid F) = \frac{P(F \mid S) \cdot P(S)}{P(F)} = \frac{0.2 \cdot 0.6}{0.36} = 0.333$$

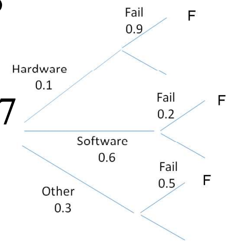


Calculate P(O|F) using Bayes rule

$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.9 \cdot 0.1}{0.36} = 0.250$$

$$P(S \mid F) = \frac{P(F \mid S) \cdot P(S)}{P(F)} = \frac{0.2 \cdot 0.6}{0.36} = 0.333$$

$$P(O \mid F) = \frac{P(F \mid O) \cdot P(O)}{P(F)} = \frac{0.5 \cdot 0.3}{0.36} = 0.417$$



Note that the conditionals given failure add to 1.

Printer failure interpretation

Because $P(O \mid F)$ is largest, the most likely cause of the problem is in the *other* category.

Screening test for disease: Bayes' Rule example

b) The proportion of people in a given community who have a certain disease is 0.005. A test is available to diagnose the disease. If a person has the disease, the probability that the test will produce a positive signal is 0.99. If a person does not have the disease, the probability that the test will produce a positive signal is 0.01. If a person tests positive, calculate the probability that the person actually has the disease? [10 marks]

Calculate probability of disease given test +

Let D represent the event that the person actually has the disease, and let + represent the event that the test gives a positive signal. We wish to find P(D|+). We are given the following probabilities:

P(D) = 0.005, P(+ | D) = 0.99, P (+ | not D) = 0.01

P(D | +) = (0.99)(0.005) / ((0.99)(0.005) + (0.01)(0.995))= 0.332

```
(0.99)*(0.005) / ( (0.99)*(0.005) + (0.01)*(0.995) )
[1] 0.3322148
```


Bayes Theorem with Total Probability

If E_1, E_2, … E_k are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \ldots + P(B \mid E_k)P(E_k)}$$

where $P(B) > 0$

Note : Numerator expression is always one of the terms in the sum of the denominator.

Example: Astronauts (Bayes' Rule)

- Astronauts on the shuttle realise that oxygen levels are dropping. There are 3 possible problems that can cause oxygen levels to drop (O): a leak in fuselage (L), malfunctioning oxygen pump (M) and a CO_2 filter in need of replacement (F). It is known that:

  P(L) = 0.02,

  P(M) = 0.49,

  P(F) = 0.49.

Ground crew run simulations to find:

  P(O | L) = 1, P(O | M) = 0.4, P(O | F) = 0.6,

What should the astronauts try to fix first ?

Calculate P(L|O)

$$P(L|O) = \frac{P(O|L)P(L)}{P(O|L)P(L)+P(O|M)P(M)+P(O|F)P(F))}$$

$$= \frac{1*0.02}{(1*0.02) + (0.4*0.49) + (0.6*0.49)}$$

$$= \frac{0.02}{0.51}$$

$$= 0.039$$

Calculate P(M|O)

$$P(M|O) = \frac{P(O|M)P(M)}{P(O|L)P(L)+P(O|M)P(M)+P(O|F)P(F))}$$

$$= \frac{0.4 * 0.49}{0.51}$$

$$= 0.384$$

Calculate P(FIO)

$$P(F|O) = \frac{P(O|F)P(F)}{P(O|L)P(L)+P(O|M)P(M)+P(O|F)P(F))}$$

$$= \frac{0.6 * 0.49}{0.51}$$

= **0.576**

P(M | O) = 0.384

P(H | O) = 0.039

Astronauts should check the filter first!

Note that P(L|O) + P(M|O) + P(F|O) = 1

Section Summary

- Sample spaces (list by hand or use counting techniques)
  - Permutations and combinations
- Probability
  - Axioms
  - Joint events as Unions (“or”) or intersections (”and”)
  - For unions: mutually exclusive events ?
  - For intersections: independent events ?
  - Conditional probability and Bayes Rule

What Can Go Wrong?

- Beware of probabilities that don’t add up to 1.
  - To be a legitimate probability distribution, the sum of the probabilities for all possible outcomes must total 1.
- Don’t add probabilities of events if they’re not disjoint.
  - Events must be disjoint to use the Addition Rule.