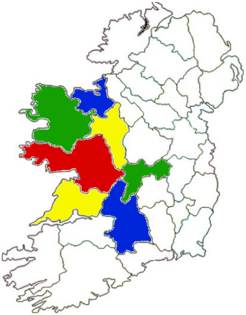
MA284 : Discrete Mathematics

Week 10: Colouring Graphs; Eulerian and Hamiltonian Graphs

Dr Kevin Jennings

9 and 11 November, 2021



See also Section 4.4 of Levin's *Discrete Mathematics.*

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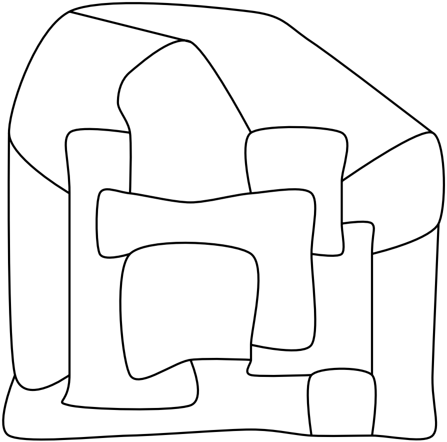
PART 1: Vertex Colouring

[From textbook, p184]. Here is a map of the (fictional) country "Euleria". Colour it so that adjacent regions are coloured differently. What is the fewest colours required?



There are maps that can be coloured with

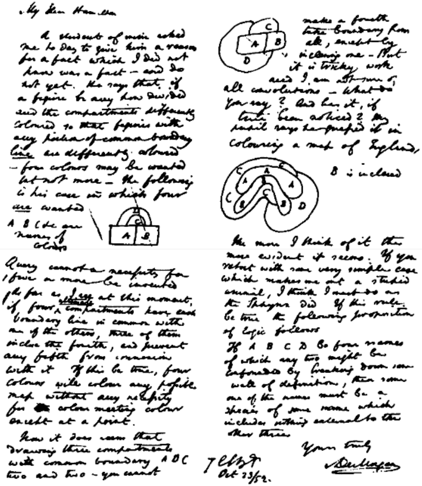
- A single colour:
- Two colours (e.g., the island of Ireland):
- Three colours:
- **Four colours:**

It turns out that the is *no* map that needs more than 4 colours. This is the famous Four Colour Theorem, which was originally conjectured by the British/South African mathematician and botanist, Francis Guthrie who at the time was a student at University College London.

He told one of his mathematics lecturers, Augustus de Morgan, who, on **23 October, 1852**, wrote to friend William Rowan Hamilton, who was in Dublin:

From `https://en.wikipedia.org/wiki/Four_color_theorem`

de Morgan writes to Hamilton, 23 October, 1852..



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary ABC two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

7 Camden St
Oct 23/52.

From https://en.wikipedia.org/wiki/Four_color_theorem
de Morgan writes to Hamilton, 23 October, 1852..

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured-four colours may be wanted, but not more-the following is his case in which four are wanted. Query: cannot a necessity for five or more be invented... What do you say? And has it, if true been noticed?

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De Morgan needn't have worried: a proof was not produced until **1976**. It is very complicated, and relies heavily on computer power.

To get a sense of *why* it might be true, try to draw a map that needs 5 colours.

. . . .

Our interest is not in trying to prove the Four Colour Theorem, but in how it is related to Graph Theory.

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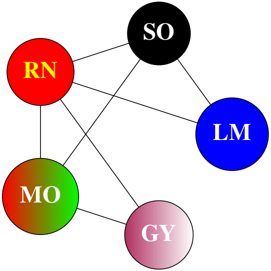
PART 2: Colouring Graphs

Our interest is not in trying to prove the Four Colour Theorem, but in how it is related to Graph Theory

If we think of a map as a way of showing which regions share borders, then we can represent it as a *graph*, where

- A vertex in the graph corresponds to a region in the map;
- There is an edge between two vertices in the graph if the corresponding regions share a border.

Example:



Colouring regions of a map corresponds to colouring vertices of the graph. Since neighbouring regions in the map must have different colours, so too adjacent vertices in the graph must have different colours.

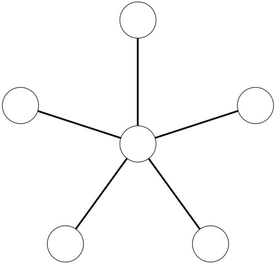
More precisely

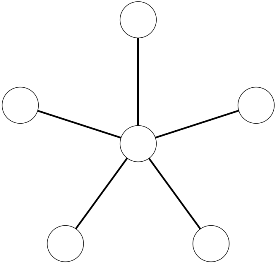
Vertex Colouring: An assignment of colours to the vertices of a graph is called a *VERTEX COLOURING*.

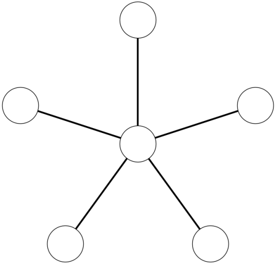
Proper Colouring: If the vertex colouring has the property that adjacent vertices are coloured differently, then the colouring is called *PROPER*.

Lots of different proper colourings are possible. If the graph has v vertices, then clearly at most v colours are needed. However, usually, we need far fewer.

Examples:







CHROMATIC NUMBER

The smallest number of colours needed to get a proper vertex colouring of a graph G is called the *CHROMATIC NUMBER* of the graph, written $\chi(G)$.

Example: Determine the Chromatic Number of the graphs C_2, C_3, C_4 and C_5.

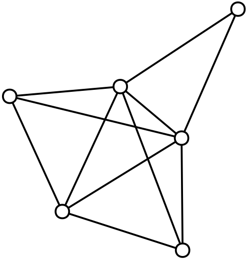
Example: Determine the Chromatic Number of the K_n and $K_{p,q}$ for any n, p, q.

In general, calculating $\chi(G)$ is not easy. There are some ideas that can help. For example, it is clearly true that, if a graph has v vertices, then

$$1 \leq \chi(G) \leq v.$$

If the graph happens to be *complete*, then $\chi(G) = v$. If it is *not* complete the we can look at *cliques* in the graph.

Clique: A *CLIQUE* is a subgraph of a graph all of whose vertices are connected to each other.



The **CLIQUE NUMBER** of a graph, G, is the number of vertices in the largest clique in G.

From the last example, we can deduce that

LOWER BOUND: The chromatic number of a graph is *at least* its clique number.

. . . .

We can also get a useful upper bound. Let $\Delta(G)$ denote the largest degree of any vertex in the graph, G,

UPPER BOUND: $\chi(G) \leq \Delta(G) + 1$.

Why? This is called **Brooks' Theorem**, and is Thm 4.5.5 in the text-book: `http://discrete.openmathbooks.org/dmoi3/sec_coloring.html`

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PART 3: Algorithms for $\chi(G)$

In general, finding a proper colouring of a graph is *hard*.

There are some algorithms that are efficient, but not optimal. We'll look at two:

1. The *Greedy algorithm*.
2. The *Welsh-Powell algorithm.*

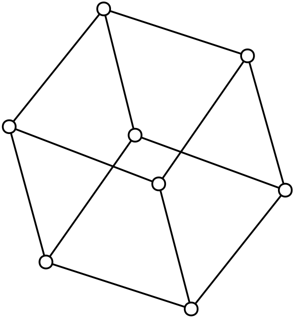
The **Greedy algorithm** is simple and efficient, but the result can depend on the ordering of the vertices.

Welsh-Powell is *slightly* more complicated, but can give better colourings.

The GREEDY ALGORITHM

1. Number all the vertices. Number your colours.
2. Give a colour to vertex 1.
3. Take the remaining vertices in order. Assign each one the lowest numbered colour, that is different from the colours of its neighbours.

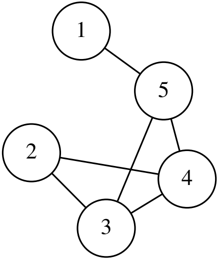
Example: Apply the GREEDY ALGORITHM to colouring the following graph (the cubical graph, Q_3):



Welsh-Powell Algorithm

1. List all vertices in decreasing order of their degree (so largest degree is first). If two or more have the same degree, list those any way.
2. Colour the first listed vertex (with first unused colour).
3. Work down the list, giving that colour to all vertices *not* connected to one previously coloured.
4. Cross coloured vertices off the list, and return to the start of the list.

Example: Colour this graph using both GREEDY and WELSH-POWELL:



Example

Seven one-hour exams, e_1, e_2, $\ldots e_7$, must be timetabled. There are students who must sit

(i) e_1 and e_5,

(ii) e_1 and e_7,

(iii) e_2, e_3, and e_6,

(iv) e_2, e_4, and e_7,

(v) e_3, e_5, and e_6,

(vi) e_4 and e_5

Model this situation as a vertex colouring problem, and find a scheduling that avoids timetable clashes and uses the minimum number of hours.

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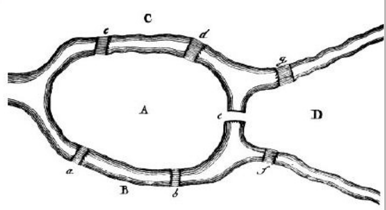
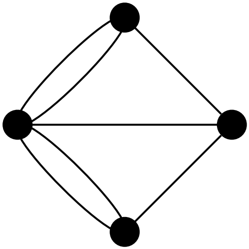
PART 4: Eulerian Paths and Circuits

We originally motivated the study of Graph Theory with the *Königsberg bridges* problem: find a route through the city that crosses every bridge once and only once:



The seven bridges of Königsberg



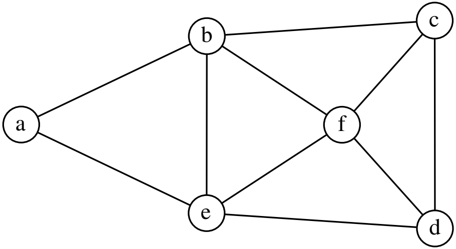
We'll now return to this problem, and show that there is no solution. First we have to re-phrase this problem in the setting of graph theory.

Recall (from Week 8) that a **PATH** in a graph is a sequence of adjacent vertices in a graph.

Eulerian Path

An **EULERIAN PATH** (also called an *Euler Path* and an *Eulerian trail*) in a graph is a path which uses every edge exactly once.

Example:

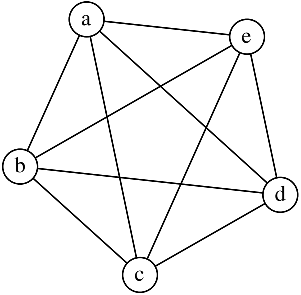


Recall from Week 8 that a **circuit** is a path that begins and ends at that same vertex, and no **edge** is repeated...

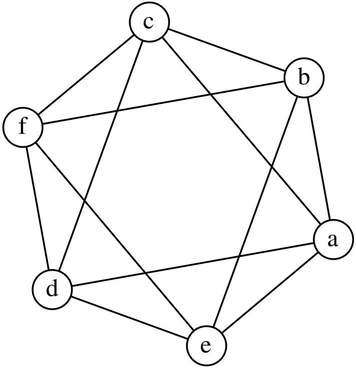
Eulerian Circuit

An **EULERIAN CIRCUIT** (also called an *Eulerian cycle*) in a graph is an *Eulerian* path that starts and finishes at the same vertex.
If a graph has such a circuit, we say it is *Eulerian*.

Example 1 (K_5):



Example 2: Find an Eulerian circuit in this graph:

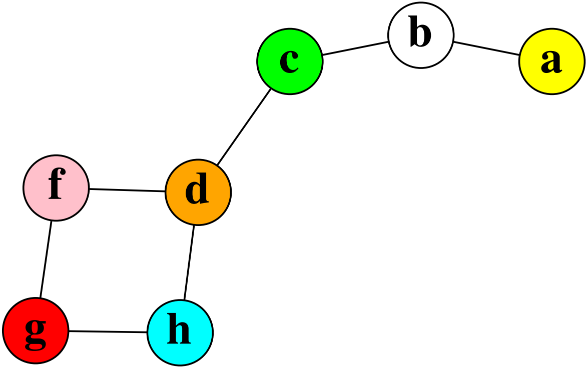


Of course, not every graph as an Eulerian circuit, or, indeed, and Eulerian path.

Here are some extreme examples:

It is possible to come up with a condition that *guarantee* that a graph has an *Eulerian path*, and, addition, one that ensures that it has an *Eulerian circuit*.

To begin with, we'll reason that the following graph could *not* have an Eulerian circuit, although it *does* have an Eulerian path:



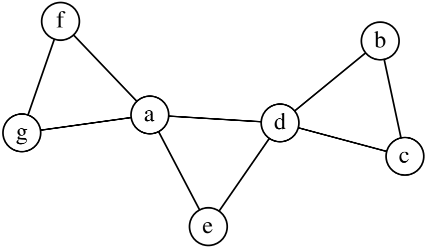
Suppose, first, the we have a graph that **does have an Eulerian circuit**. Then for every edge in the circuit that "*exits*" a vertex, there is another that "*enters*" that vertex. So every vertex must have even degree.

Example (W3)

In fact, a stronger statement is possible.

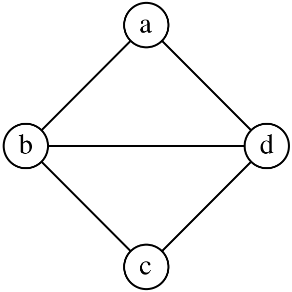
A graph has an **EULERIAN CIRCUIT** if and only if every vertex has even degree.

Example: Show that the following graph has an *Eulerian circuit*



Next suppose that a graph **does not have an Eulerian circuit**, but does have an **Eulerian Path**. Then the degree of the "start" and "end" vertices must be odd, and every other vertex has even degree.

Example:

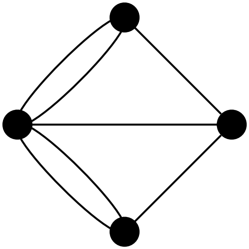


To summarise:

Eulerian Paths and Circuits

- A graph has an **EULERIAN CIRCUIT** *if and only if* the degree of every vertex is even.
- A graph has an **EULERIAN PATH** *if and only if* it has either **zero** or **two** vertices with odd degree.

Example: The *Königsberg bridge* graph does not have an Eulerian path:



Example (MA284, 2020/21 Semester 1 Exam)

Let $G = (V, E)$, where $V = \{a, b, c, d, e, f, g\}$, and
$E = \{\{a, b\}, \{a, g\}, \{b, c\}, \{b, d\}, \{b, g\}, \{c, d\}, \{d, e\}, \{e, f\}, \{e, g\}, \{f, g\}$.
Does G admit an Eulerian Path and/or Circuit? If it does, exhibit one. If not, explain why.

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PART 5: Hamiltonian Paths and Cycles

Next we'll look at a closely related problem: finding paths through a graph that visit every vertex exactly once.

These are called **HAMILTONIAN PATH**, and are named after the (very famous) William Rowan Hamilton, the Irish mathematician, who invented a board-game based on the idea.

Hamilton's Icosian Game (Library of the Royal Irish Academy)

Try playing online: `https://www.geogebra.org/m/u3xggkcj`

Definition (HAMILTONIAN PATH)

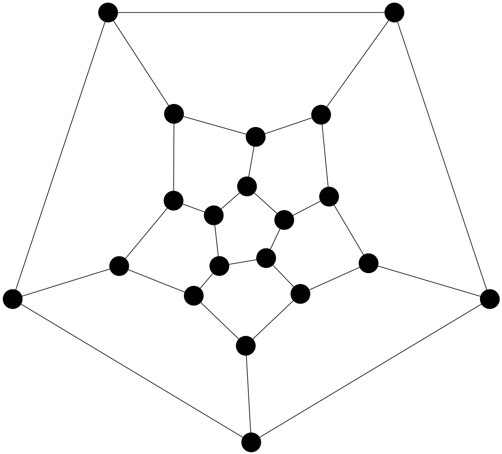
A path in a graph that visits every vertex exactly once is called a **HAMILTONIAN PATH**.

Hamiltonian Cycles

Recall that a **CYCLE** is a path that starts and finishes at the same vertex, but no other vertex is repeated.
A **HAMILTONIAN CYCLE** is a cycle which visits the start/end vertex twice, and every other vertex exactly once.
A graph that has a Hamiltonian cycle is called a **HAMILTONIAN GRAPH**.

Examples:

This is the graph based on Hamilton's Icosian game. We'll find a Hamilton path. Can you find a Hamilton cycle?



Important examples of Hamiltonian Graphs include:

- *cycle graphs*;
- *complete graphs*;
- *graphs of the platonic solids*.

In general, the problem of finding a Hamiltonian path or cycle in a large graph is **hard** (it is known to be NP-complete). However, there are two relatively simple *sufficient conditions* to testing if a graph is Hamiltonian.

1. Ore's Theorem

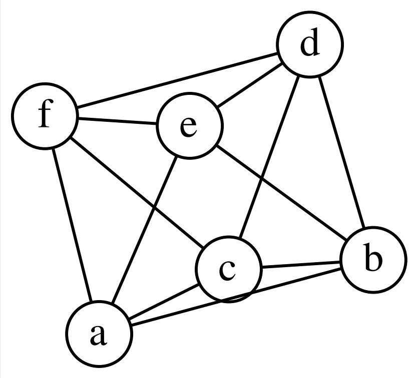
A graph with v vertices, where $v \geq 3$, is *Hamiltonian* if, for every pair of non-adjacent vertices, the sum of their degrees $\geq v$.

2. Dirac's Theorem

A (simple) graph with v vertices, where $v \geq 3$, is *Hamiltonian* if every vertex has degree $\geq v/2$.

Example

Determine whether or not the graph illustrated below is Hamiltonian, and if so, give a Hamiltonian cycle:



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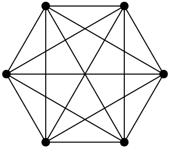
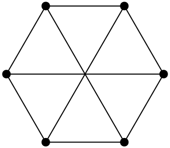
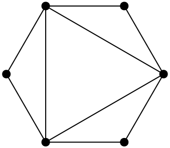
END OF PART 5

Q1. (Textbook) What is the smallest number of colors you need to properly color the vertices of $K_{4,5}$? That is, find the chromatic number of the graph.

Q2. Determine the chromatic number of each of the following graphs, and give a colouring for that achieves it.

(i) 

(ii) 

(iii) 

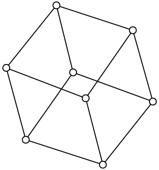
Q3. For each of the following graphs, determine if it has an Eulerian path and/or circuit. If not, explain why; otherwise give an example.

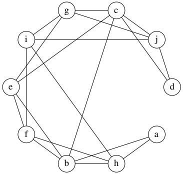
(a) K_n, with n even.

(b) $G_1 = (V_1, E_1)$ with $V_1 = \{a, b, c, d, e, f\}$,
$E_1 = \{\{a, b\}, \{a, f\}, \{c, b\}, \{e, b\}, \{c, e\}, \{d, c\}, \{d, e\}, \{b, f\}\}$.

(c) $G_2 = (V_2, E_2)$ with $V_2 = \{a, b, c, d, e, f\}$,
$E_2 = \{\{a, b\}, \{a, f\}, \{c, b\}, \{e, b\}, \{c, e\}, \{d, c\}, \{d, e\}, \{b, f\}, \{b, d\}\}$.

Q4. For each of the following graphs, determine if it has an *Eulerian path* and/or *Eulerian circuit*. If so, give an example; if not, explain why.

$G_1 =$ 

$G_2 =$ 

Q5. *Given a graph $G = (V, E)$, its **compliment** is the graph that has the same vertex set, V, but which has an edge between a pair of vertices **if and only if** there is no edge between those vertices in G.*

Sketch of of the following graphs, and their complements:

(i) K_4, (ii) C_4, (iii) P_4, (iv) P_5.

Q6. Which of the following graphs are isomorphic to their own complement ("self-complementary")?

(i) K_4, (ii) C_4, (iii) P_4, (iv) P_5.

Q7. Show that $K_{3,3}$ has Hamiltonian, but $K_{2,3}$ is not.