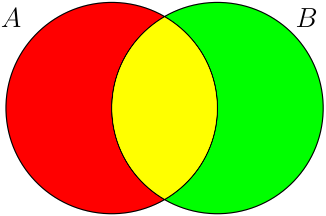
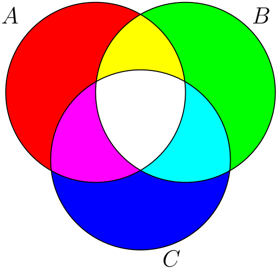
MA284 : Discrete Mathematics

Week 2: Counting with sets and the PIE

Dr Kevin Jennings

14 & 16 September, 2022





Tutorials will start next week (week beginning Monday, 19 September).

You should attend *one tutorial per week*.

The proposed tutorial times are

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			MRA 201		(Lecture)
12 – 1		CA117			
1 – 2			(Lecture)		
2 – 3			AC215		
3 – 4		AC213		ADB1020	
4 – 5				AMB-G008	

Please email Kevin now if *none* of these times work for you, with your course details.

SUMS also opens next week. Dr Kirsten Pfeiffer will be here next week to give more information.

We will use WeBWorK for all assignments in this module. You can access them by logging on to Blackboard, clicking on `Assignments`, and then the relevant link.

At present (14 Sep) , there is just a `Demo Assignment` there. Please try it out, and report any problems. There are 10 questions, and you may attempt each one up to 10 times.
This problem set does **not** contribute to your CA score for MA284.

The first proper assignment will open on Friday.

In this week's classes, we are going to build on the *Additive* and *Multiplicative* Principles from Lecture 2.
After reminding ourselves of the basic ideas, we will present them in the formal setting of *set theory*.

We will then move on to the *Principle of Inclusion/Exclusion* (PIE).
The presentation will closely follow Chapter 1 of Levin's *Discrete Mathematics: an open introduction*.

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Start of ...

PART 1: Review from Week 1

The Additive Principle

If Event A can occur m ways, and Event B can occur n (disjoint) ways, then Event "A **or** B" can occur in $m + n$ ways.

Example

There are (now) **235** students in registered for Discrete Mathematics, of which **60** are in Financial Maths & Economics (FM), **55** are in Arts, and the remaining **120** are in various Sciences (including Computer Science).

1. In how many ways can we choose a Class Rep who is from **Arts** or **FM**?
2. How many ways can be chosen a Class Rep who is from **Arts**, **FM**, or **Science**?

The Multiplicative Principle

If Event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then Event "A **and** B" can occur in $m \times n$ ways.

Example

There are (still) **235** students in registered for Discrete Mathematics, of which **60** are in Financial Maths & Economics (FM), **55** are in Arts, and the remaining **120** are in various Sciences (including Computer Science).

1. In how many ways can we choose two Class Reps, one each from **Arts** and **FM**?
2. How many ways can we choose three Class Reps, one each from **Arts**, **FM**, and **Science**?

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END OF PART 1

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PART 2: Counting with Sets

Example (Students in Discrete Mathematics (again))

Let D be the set of students in Discrete Mathematics. So $|D| = 235$.
Let F be the set of Discrete Maths students who are in **Financial Maths**. So $|F| = 60$.
Similarly, let S and A be the sets of Discrete Mathematics students who are in **Science** and **Arts** respectively. So $|S| = 120$, and $|A| = 55$.

What do we mean by...

- $A \cup F$?
- $A \cap F$?

Additive Principle in terms of "events"

If Event A can occur m ways, and Event B can occur n (disjoint/ independent) ways, then event "A **or** B" can occur in $m + n$ ways.

But an "event" can be expressed as just selecting an element of a set. For example, the event "*Choose a Class Rep from Arts*" is the same as "*Choose an element of the set A*". Similarly:

- **Event A can occur m ways**, is the same as saying $|A| = m$;
- **Event B can occur n ways**, is the same as saying $|B| = n$;
- Events A and B are disjoint/independent means $|A \cap B| = 0$ (or, equivalently $A \cap B = \emptyset$).

Additive Principle for Sets

Given two sets A and B with $|A| = m$, $|B| = n$ and $|A \cap B| = 0$. Then

$$|A \cup B| = m + n.$$

Additive Principle for Sets

Given two sets A and B with $|A \cap B| = 0$. Then

$$|A \cup B| = |A| + |B|.$$

Example:

The **Cartesian Product** of sets A and B is

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

This is the set of pairs where the first term in each pair comes from A, **and** the second comes from B.

Example

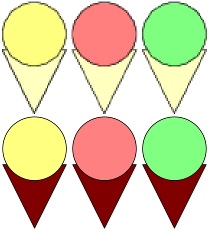
Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4\}$.
Write down $A \times B$ and $A \times C$.

If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.
Why?

What has the *Cartesian Product* got to do with the **Multiplicative Principle**? Consider the following example... Suppose we go to our favourite ice-cream shop where they stock

- three flavours: **V**anilla, **Strawberry** and **M**int.
- two types of cone: plain **C**ones and **W**affle cones.

How many ways can I place an order (for 1 cone and 1 scoop?).



Previously we learned about

The Multiplicative Principle (for events)

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

We can now express this in terms of sets:

Multiplicative Principle for Sets

Given two sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

This extends to three or more sets in the obvious way:

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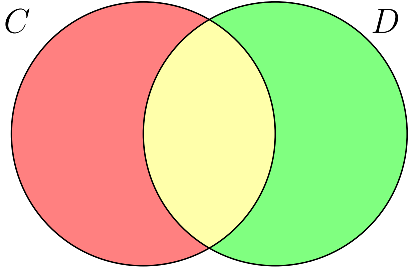
PART 3: The Principle of Inclusion and Exclusion (PIE)

Good news!

Remember from last week that the NUIG Animal Shelter had 4 cats and 6 dogs in need of a home. Well, they have all been adopted and, (unsurprisingly, given their kind and generous nature) by Discrete Mathematics students. They went to 9 different homes, because one person adopted both a cat and a dog.



Since we admire those people that adopted an animal so much, we want one of them as our Class Rep. That is we will choose our Class Rep from one of the sets C and D where $|C| = 4$ and $|D| = 6$.

If we were to apply the **Additive Principle** *naïvly*, we would think that we have $|C| + |D| = 10$ choices for our Rep. But of course, we only have $|C \cup D| = 9$ choices.

So, to correctly calculate the cardinality of a pair of sets (with non-zero intersection) we need *the Principle of Inclusion and Exclusion*.

The Principle of Inclusion and Exclusion (for the union of 2 sets)

For any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This extends to larger numbers of sets. For example,

The Principle of Inclusion and Exclusion, for the union of 3 sets

For any finite sets A, B, and C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example (PIE for 2 sets)

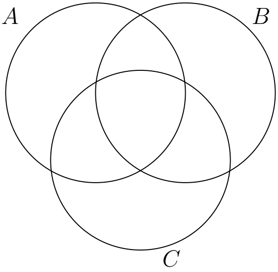
A group of 20 second year maths students are registering for modules. 12 take Discrete Mathematics and, of those, 4 take both Discrete Maths and Differential Forms. If all 20 do at least one of these subjects, how many just take Differential Forms?

Example (See Example 1.1.8 of textbook)

An examination in three subjects, **A**lgebra, **B**iology, and **C**hemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations.

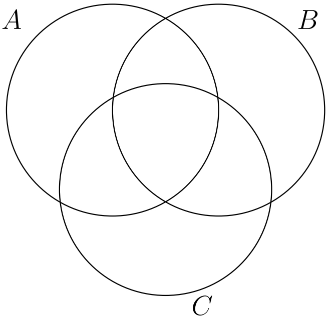
Subject:	A	B	C	A&B	A&C	B&C	A&B&C
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?



This example shows how to extend the PIE to three sets:

$$
\begin{aligned}
|A \cup B \cup C| = |A| + |B| + |C| \\
- |A \cap B| - |A \cap C| - |B \cap C| \\
+ |A \cap B \cap C| \qquad .
\end{aligned}
$$



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PART 4: Subsets & Power Sets

Start here Friday, 16 September

Recall last week it was mentioned that one of the earliest recorded problems in combinatorics is from the *Sushruta Samhita* an ancient Sanskrit text on medicine and surgery.

Palm leaves of the Sushruta Samhita or Sahottara-Tantra from Nepal. Source: https://en.wikipedia.org/wiki/Sushruta_Samhita

The **combinatorics** problem from the Sushruta Samhita is to determine the number of **different possible combinations** of the tastes

(1) *sweet* (2) *pungent* (3) *astringent* (4) *sour* (5) *salt* and (6) *bitter*.

This is equivalent to the problem of *counting the number of non-empty subsets* there are of a set with 6 elements.

The question we will investigate is:

How many subsets are there of $A_1 = \{1\}$?

How many subsets are there of $A_2 = \{1, 2\}$?

How many subsets are there of $A_3 = \{1, 2, 3\}$?

How many subsets are there of $A_4 = \{1, 2, 3, 4\}$?

$\vdots$

How many subsets are there of $A_k = \{1, 2, 3, \ldots, k\}$?

Here is another way of expressing this:

Power set

The POWER SET of A, denoted by $P(A)$, is the set of all subsets of A, including the empty set.
What is $|P(A)|$?

We'll answer this question in two different ways, which is a classic approach to problems in combinatorics.

First we'll list all the subsets of A_1, A_2 and A_3, and try to guess the answer. Then we will try to explain it.

Here is another approach. Consider $P(A_2) = P(\{1, 2\})$.
When constructing a subset, we can proceed as follows:

- **Event A:** choose to include the element **1** or not. This can happen in 2 ways.
- **Event B:** choose to include the element **2** or not. This can happen in 2 ways.

Now apply the multiplicative principle.

Example

How many subsets are there of $A_5 = \{1, 2, 3, 4, 5\}$?

Here is a slightly harder problem

How many subsets are there of $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

We will look at **three** different ways of answering this question:

1. By "brute-force": simply listing all the possibilities.
2. By counting all sets that **do not** have three elements.
3. Next week, by using **binomial coefficients**.

Method 2

How many subsets are there of $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

Here is an easy way of answering this question.

- How many subsets of A_5 have no elements?
- How many subsets of A_5 have 5 elements?
- How many subsets of A_5 have 1 element?
- How many subsets of A_5 have 4 elements?
- Now use that the number of subsets of A_5 with **3** elements, is the same as the number with **2** elements.

Here are a set of exercises to help you work through the material presented during Week 2.
Except where indicated, all these exercises are taken from Section 1.1 of textbook (Levin's Discrete Mathematics - an open introduction).

1 We usually write numbers in decimal form (i.e., base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes, though, it is useful to write numbers in *hexadecimal* (base 16), which has 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

a. How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).
b. Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?
c. How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.
d. How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

2 A group of students were asked about their TV watching habits. Of those surveyed,

- 28 students watch *The Good Place*,
- 19 watch *Stranger Things*, and
- 24 watch *Orange is the New Black*.

- Additionally, 16 watch *The Good Place* and *Stranger Things*,
- 14 watch *The Good Place* and *Orange is the New Black*,
- and 10 watch *Stranger Things* and *Orange is the New Black*.
- There are 8 students who watch all three shows.

How many students surveyed watched at least one of the shows?

3 (MA284, Final Exam, 2018/2019) In a survey, 36 students were asked if they liked Discrete Mathematics, Statistics and Differential Forms. 16 said they liked Discrete Maths, 20 liked Statistics, 26 admitted to liking Differential Forms, and 1 did not like any.
Additionally, 9 students said they liked both Discrete Maths and Statistics, 13 liked Statistics and Differential Forms, and 11 liked Discrete Maths and Differential Forms. How many students like *all* three subjects?

4 In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students do *not* like potatoes? Explain why your answer is correct.

5 (MA284, Semester 1 Exam, 2016/2017) For how many $n \in \{1, 2, \dots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

6 Let A, B, and C be sets.

a. Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

b. Describe a set in terms of A, B, and C with cardinality 26.

7 (MA284, Semester 1 Exam, 2017/2018) The sets A and B are such that $|A| = 17$ and $|B| = 9$.
What is the largest possible value of $|A \cup B|$?
What is the smallest possible value of $|A \cup B|$?
What is the largest possible value of $|A \cap B|$?
What is the smallest possible value of $|A \cap B|$?
What is the value of $|A \cup B| + |A \cap B|$?